5

LIMITS, ANALYTIC GEOMETRY, AND APPROXIMATIONS

5.1 INFINITE LIMITS

Up to this point we have studied three types of limits:

$$\lim_{x\to c} f(x) = L \quad \text{means} \quad f(x) \approx L \text{ whenever } x \approx c \text{ but } x \neq c.$$

$$\lim_{x\to c^+} f(x) = L \quad \text{means } f(x) \approx L \text{ whenever } x \approx c \text{ but } x > c.$$

$$\lim_{x\to c^-} f(x) = L \quad \text{means } f(x) \approx L \text{ whenever } x \approx c \text{ but } x < c.$$

The limit notation $\lim_{x\to\infty} f(x) = L$ means that whenever H is positive infinite, $f(H) \approx L$ (Figure 5.1.1(a)).

$\lim_{x\to c} f(x) = -\infty$ means that whenever $x \approx c$ and $x \neq c$, $f(x)$ is negative infinite (Figure 5.1.1(b)). The various other combinations have the meanings which one would expect.

EXAMPLE 1 $\displaystyle\lim_{x\to 0}\frac{1}{x^2} = \infty.$

EXAMPLE 2 $\displaystyle\lim_{x\to 0^+}\frac{1}{x} = \infty, \qquad \lim_{x\to 0^-}\frac{1}{x} = -\infty.$

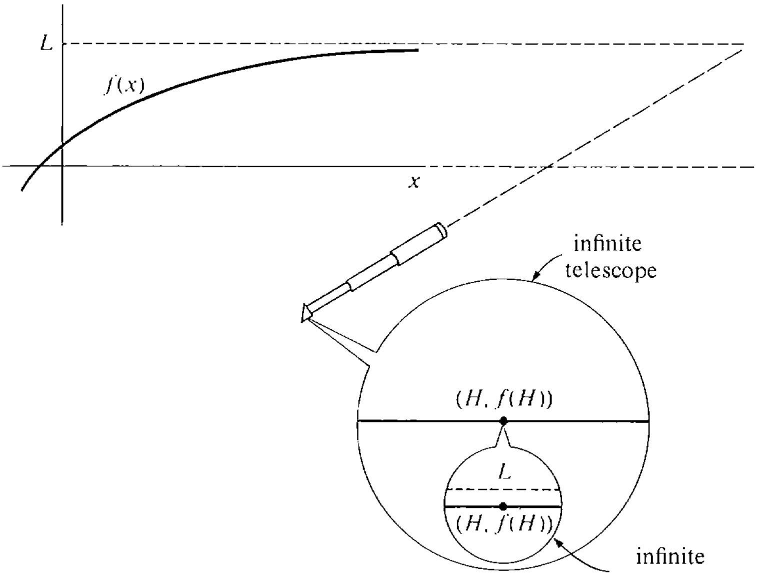
EXAMPLE 3 Find $\displaystyle\lim_{x\to\infty}\frac{3x^4 + 5x - 2}{2x^4 - 6x^3 + 7}.$

Let H be positive infinite. Then

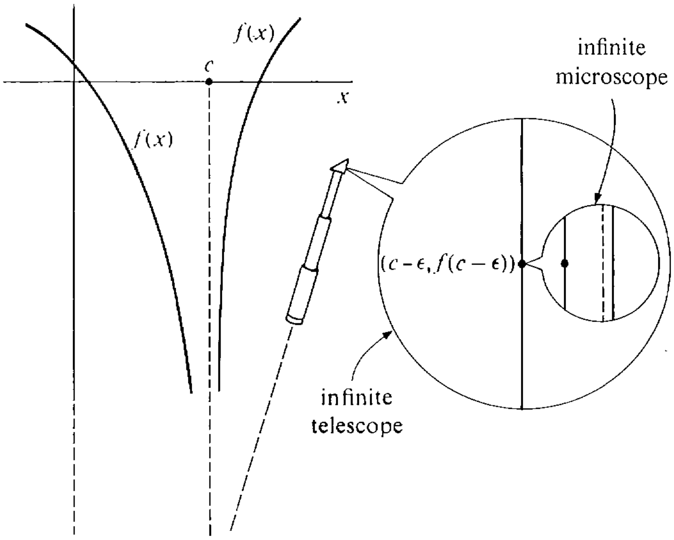
$$\frac{3H^4 + 5H - 2}{2H^4 - 6H^3 + 7} = \frac{3 + 5H^{-3} - 2H^{-4}}{2 - 6H^{-1} + 7H^{-4}},$$

and therefore $$st\left(\frac{3H^4 + 5H - 2}{2H^4 - 6H^3 + 7}\right) = \frac{3 + 0 - 0}{2 - 0 + 0} = \frac{3}{2}.$$

Thus the limit exists and is $\frac{3}{2}$.



(a) $\lim_{x \to \infty} f(x) = L$



(b) $\lim_{x \to c} f(x) = -\infty$

Figure 5.1.1

EXAMPLE 4 Find $\lim_{x \to -\infty} (x^3 + 200x^2)$.

We have $x^3 + 200x^2 = x^2(x + 200)$. When H is negative infinite, H^2 is positive infinite and $(H + 200)$ is negative infinite, so their product is negative infinite. Thus

$$\lim_{x \to -\infty} (x^3 + 200x^2) = -\infty.$$

When $\lim_{x \to c} f(x) = \infty$ or $-\infty$,

the limit does not exist, because $f(x)$ has no standard part. The infinity symbol is only used to indicate the behavior of $f(x)$ and is not to be construed as a number.

EXAMPLE 5 A student can get a score of $100t/(t+1)$ on his math exam if he studies t hours for it (Figure 5.1.2). If he studies infinitely long for the exam, his score will be infinitely close to 100, because if H is positive infinite,

$$st\left(\frac{100H}{H+1}\right) = st\left(\frac{100}{1+1/H}\right) = \frac{100}{1+0} = 100.$$

In the notation of limits,

$$\lim_{t \to \infty} \frac{100t}{t+1} = 100.$$

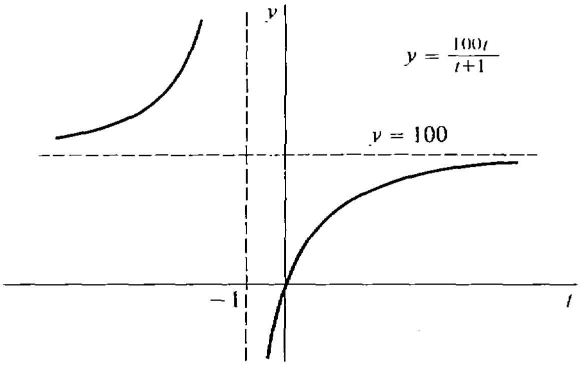


Figure 5.1.2

EXAMPLE 6 Given any polynomial

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

of degree $n > 0$, the limits as t approaches $-\infty$ or $+\infty$ are as follows. Suppose $a_n > 0$. When n is even, $\lim_{t \to -\infty} f(t) = \infty$, $\lim_{t \to \infty} f(t) = \infty$. When n is odd, $\lim_{t \to -\infty} f(t) = -\infty$, $\lim_{t \to \infty} f(t) = \infty$.

The signs are all reversed when $a_n < 0$.

All these limits can be computed from

$$f(t) = t^n\left(a_n + \frac{a^{n-1}}{t} + \cdots + \frac{a_1}{t^{n-1}} + \frac{a_0}{t^n}\right).$$

EXAMPLE 7 In the special theory of relativity, a body which is moving at constant velocity v, $-c < v < c$, will have mass

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

and its length in the direction of motion will be

$$l = l_0\sqrt{1 - v^2/c^2}.$$

Here m_0, l_0, and c are positive constants denoting the mass at rest (that is, the mass when $v = 0$), the length at rest, and the speed of light.

Suppose the velocity v is infinitely close to the speed of light c, that is,

$$v = c - \varepsilon, \qquad \varepsilon > 0 \text{ infinitesimal.}$$

Then
$$\sqrt{1 - \frac{v^2}{c^2}} = \sqrt{1 - \frac{(c-\varepsilon)^2}{c^2}} = \sqrt{\frac{c^2 - (c^2 - 2c\varepsilon + \varepsilon^2)}{c^2}}$$
$$= \sqrt{\frac{2\varepsilon}{c} - \frac{\varepsilon^2}{c^2}} = \sqrt{\varepsilon \cdot \left(\frac{2}{c} - \frac{\varepsilon}{c^2}\right)},$$

which is the square root of a positive infinitesimal. Thus $\sqrt{1 - v^2/c^2}$ is a positive infinitesimal. Therefore for v infinitely close to c, m is positive infinite and l is positive infinitesimal. That is, a body moving at velocity infinitely close to (but less than) the speed of light has infinite mass and infinitesimal length in the direction of motion. In the notation of limits this means that

$$\lim_{v \to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = +\infty,$$

$$\lim_{v \to c^-} l_0 \sqrt{1 - v^2/c^2} = 0.$$

Caution: This example must be understood in the light of our policy of speaking as if a line in physical space really is like the hyperreal line. Actually, there is no evidence one way or the other on whether a line in space is like the hyperreal line, but the hyperreal line is a useful model for the purpose of applications.

EXAMPLE 8 Evaluate $\lim\limits_{x \to \infty} \dfrac{\sin x}{x}$.

When H is positive infinite, $\sin H$ is between -1 and 1 and thus finite, so $(\sin H)/H$ is infinitesimal. The limit is therefore zero:

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0.$$

EXAMPLE 9 Find $\lim\limits_{x \to \infty} \cos x$.

If H is any integer or hyperinteger, then

$$\cos(2\pi H) = 1, \qquad \cos(2\pi H + \pi) = -1.$$

In fact, $\cos x$ will keep oscillating between 1 and -1 even for infinite x. Therefore the limit does not exist.

Limits involving e^x and $\ln x$ will be studied in Chapter 8.

PROBLEMS FOR SECTION 5.1

Find the following limits. Your answer should be a real number, ∞, $-\infty$, or "does not exist."
With a calculator, compute some values as x approaches its limit, and see what happens.

1 $\lim\limits_{x\to\infty} \dfrac{6x-4}{2x+5}$

2 $\lim\limits_{x\to-\infty} \dfrac{3x}{4x-10}$

3 $\lim\limits_{t\to\infty} t^3 - 10t^2 - 6t - 2$

4 $\lim\limits_{t\to-\infty} 4t^2 + 6t + 2$

5 $\lim\limits_{x\to\infty} \dfrac{x^2-x+4}{3x^2+2x-3}$

6 $\lim\limits_{x\to\infty} \dfrac{2x^2-4x+1}{3x^2+5x-6}$

7 $\lim\limits_{y\to-\infty} \dfrac{5y^3+3y^2+2}{3y^3-6y+1}$

8 $\lim\limits_{y\to\infty} \dfrac{y^4-y^3+1}{2y^4-4y^2+5}$

9 $\lim\limits_{x\to\infty} \dfrac{\sqrt{x+2}}{\sqrt{3x+1}}$

10 $\lim\limits_{u\to\infty} \dfrac{3+2\sqrt{u}}{4-\sqrt{u}}$

11 $\lim\limits_{x\to\infty} x - \sqrt{x}$

12 $\lim\limits_{x\to\infty} \sqrt{x} + \sqrt{x+1}$

13 $\lim\limits_{x\to\infty} \sqrt[3]{x+2}$

14 $\lim\limits_{x\to-\infty} \sqrt{2-x}$

15 $\lim\limits_{x\to\infty} \dfrac{1}{\sqrt[3]{x}}$

16 $\lim\limits_{x\to0^+} \dfrac{1}{\sqrt[3]{x}}$

17 $\lim\limits_{x\to0^-} 1 + \dfrac{1}{x}$

18 $\lim\limits_{x\to0^+} 1 + \dfrac{1}{x}$

19 $\lim\limits_{x\to0} \dfrac{1}{x^2} - \dfrac{1}{x}$

20 $\lim\limits_{x\to0^+} \dfrac{1}{\sqrt{x}} - \dfrac{1}{x}$

21 $\lim\limits_{x\to\infty} \dfrac{5x+6}{x^2-4}$

22 $\lim\limits_{x\to\infty} \dfrac{10x^2+x+2}{x^3-4x^2-1}$

23 $\lim\limits_{t\to\infty} \dfrac{t}{\sqrt{4t^2+1}}$

24 $\lim\limits_{t\to-\infty} \dfrac{t}{\sqrt{4t^2+1}}$

25 $\lim\limits_{t\to-\infty} \dfrac{\sqrt{t^2+2}}{4t+2}$

26 $\lim\limits_{t\to\infty} \dfrac{\sqrt{t^2+2}}{4t+2}$

27 $\lim\limits_{t\to\infty} \dfrac{5t+2}{t^2-6t+1}$

28 $\lim\limits_{t\to\infty} \dfrac{t^3-6t^2+4}{2t^4+t^3-5}$

29 $\lim\limits_{t\to0} \dfrac{1-5t^{-1}}{4+6t^{-1}}$

30 $\lim\limits_{t\to0} \dfrac{5+6t^{-1}+t^{-2}}{8-3t^{-1}+2t^{-2}}$

31 $\lim\limits_{t\to0} \dfrac{1+2t^{-1}}{7+t^{-1}-5t^{-2}}$

32 $\lim\limits_{t\to0} \dfrac{1-2t^{-1}+t^{-2}}{3-4t^{-1}}$

33 $\lim\limits_{x\to2} \dfrac{1-x}{2-x}$

34 $\lim\limits_{x\to2^+} \dfrac{1-x}{2-x}$

35 $\lim\limits_{y\to3^+} \dfrac{y+1}{(y-2)(y-3)}$

36 $\lim\limits_{y\to3^-} \dfrac{y+1}{(y-2)(y-3)}$

37 $\lim\limits_{y\to3} \dfrac{y+1}{(y-2)(y-3)}$

38 $\lim\limits_{x\to5} \dfrac{3x^2+4}{x^2-10x+25}$

39 $\lim\limits_{t\to1} \dfrac{3t^3+4}{t^2+t-2}$

40 $\lim\limits_{x\to2} \dfrac{x^2+4}{x^2-4}$

41 $\lim\limits_{x\to2^+} \dfrac{x^2+4}{x^2-4}$

42 $\lim\limits_{x\to1^+} \dfrac{x-1}{x-2\sqrt{x}+1}$

43 $\lim\limits_{x\to\infty} \sqrt{x+2} - \sqrt{x}$

44 $\lim\limits_{x\to\infty} (x+1)^{3/2} - x^{3/2}$

45 $\lim\limits_{x\to\infty} \sqrt{3x+1} - 2\sqrt{x}$

46 $\lim\limits_{x\to\infty} \sqrt{2x+3} - \sqrt{x}$

47 $\lim_{x\to\infty} \sqrt{x^2+x} - x$

48 $\lim_{x\to\infty} \sqrt{x^2+1} - x$

49 $\lim_{t\to\infty} t(\sqrt{t+1} - \sqrt{t})$

50 $\lim_{t\to\infty} \sqrt{t}(\sqrt{t+2} - \sqrt{t+1})$

51 $\lim_{u\to\infty} \sqrt{u^2-3u+2} - \sqrt{u^2+1}$

52 $\lim_{x\to\infty} \dfrac{\sqrt{x+2}-\sqrt{x}}{\sqrt{x+3}-\sqrt{x}}$

53 $\lim_{t\to\infty} \sqrt[3]{t+4} - \sqrt[3]{t}$

54 $\lim_{t\to\infty} \sqrt[3]{t^3+1} - t$

55 $\lim_{t\to\infty} \cos(1/t)$

56 $\lim_{t\to\infty} \sin(1/t)$

57 $\lim_{t\to\infty} \dfrac{\sin t}{t}$

58 $\lim_{t\to\infty} \dfrac{\cos t}{t^2}$

59 $\lim_{\theta\to\infty} \sin\theta$

60 $\lim_{\theta\to\infty} \theta\cos\theta$

61 $\lim_{\theta\to 0} \tan\theta$

62 $\lim_{\theta\to\pi/2} \tan\theta$

63 $\lim_{\theta\to\pi/2^+} \tan\theta$

64 $\lim_{\theta\to\pi/2^-} \tan\theta$

65 $\lim_{x\to 0} \sin(1/x)$

66 $\lim_{x\to 0} x\cos(1/x)$

67 $\lim_{x\to 0^+} \dfrac{\cos x}{x}$

68 $\lim_{x\to 0^-} \dfrac{\cos x}{x}$

69 Prove that if $\lim_{x\to c} f(x) = \infty$ then $\lim_{x\to c} 1/f(x) = 0$.

71 Prove that if $\lim_{x\to c} f(x) = 0$ and $f(x) > 0$ for all x, then $\lim_{x\to c} 1/f(x) = \infty$.

72 Prove that if $\lim_{x\to 0^+} f(x)$ exists or is infinite, then

$$\lim_{x\to 0^+} f(x) = \lim_{t\to\infty} f(1/t).$$

73 Prove that if $\lim_{x\to\infty} f(x)$ exists or is infinite then

$$\lim_{x\to\infty} f(x) = \lim_{t\to 0^+} f(1/t).$$

5.2 L'HOSPITAL'S RULE

Suppose f and g are two real functions which are defined in an open interval containing a real number a, and we wish to compute the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}.$$

Sometimes the answer is easy. Assume that the limits of $f(x)$ and $g(x)$ exist as $x \to a$,

$$\lim_{x\to a} f(x) = L, \qquad \lim_{x\to a} g(x) = M.$$

If $M \neq 0$, then the limit of the quotient is simply the quotient of the limits,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

This is because for any infinitesimal $\Delta x \neq 0$,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = st\left(\frac{f(a+\Delta x)}{g(a+\Delta x)}\right) = \frac{st(f(a+\Delta x))}{st(g(a+\Delta x))} = \frac{L}{M}.$$

If $L \neq 0$ and $M = 0$, then the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

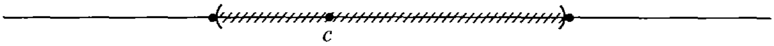
does not exist, because when $\Delta x \neq 0$ is infinitesimal, $f(a + \Delta x)$ has standard part $L \neq 0$ and $g(a + \Delta x)$ has standard part 0.

But what happens if both L and M are 0? In some cases a simple algebraic manipulation will enable us to compute the limit. For example,

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x + 1)(x - 1)}{x + 1} = \lim_{x \to -1} (x - 1) = -2,$$

even though both the numerator $x^2 - 1$ and the denominator $x + 1$ approach 0 as x approaches -1.

In other cases l'Hospital's Rule is useful in computing limits of quotients where both L and M are 0. Before stating l'Hospital's Rule, we introduce the notion of a neighborhood of a point c (Figure 5.2.1).



A neighborhood of c

Figure 5.2.1

DEFINITION

By a ***neighborhood*** *of a real number c we mean an interval which contains c as an interior point.*

The set formed by removing the point c from a neighborhood I of c is called a ***deleted neighborhood*** *of c. Thus a deleted neighborhood is the set of all points x in I such that $x \neq c$.*

L'HOSPITAL'S RULE FOR 0/0

Suppose that in some deleted neighborhood of a real number c, $f'(x)$ and $g'(x)$ exist and $g'(x) \neq 0$. Assume that

$$\lim_{x \to c} f(x) = 0, \qquad \lim_{x \to c} g(x) = 0.$$

If $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists or is infinite, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

(See Figure 5.2.2.) Usually the limit will be given by

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = \frac{f'(c)}{g'(c)},$$

and in this case the proof is very simple.

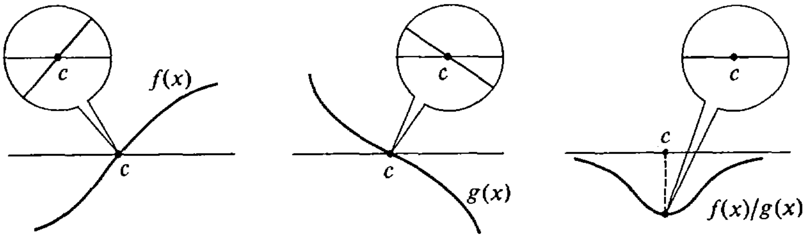


Figure 5.2.2 L'Hospital's Rule

PROOF IN THE CASE $\lim\limits_{x\to c}\dfrac{f'(x)}{g'(x)}=\dfrac{f'(c)}{g'(c)}$

Let Δx be a nonzero infinitesimal. Then $f(c)=0$, $g(c)=0$, and

$$\frac{f(c+\Delta x)}{g(c+\Delta x)}=\frac{(f(c+\Delta x)-f(c))/\Delta x}{(g(c+\Delta x)-g(c))/\Delta x}\approx\frac{f'(c)}{g'(c)}.$$

Taking standard parts we get

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{f'(c)}{g'(c)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Intuitively, for $x\approx c$ the graphs of $f(x)$ and $g(x)$ are almost straight lines of slopes $f'(c), g'(c)$ passing through zero, so the graph of $f(x)/g(x)$ is almost the horizontal line through $f'(c)/g'(c)$ (Figure 5.2.3).

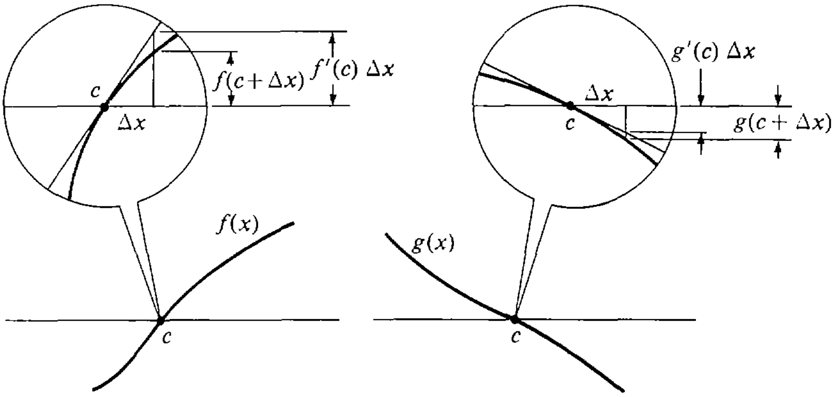


Figure 5.2.3

The equation

$$\lim_{x\to c}\frac{f'(x)}{g'(x)}=\frac{f'(c)}{g'(c)}$$

is not always true. For example, $g'(c)$ might be zero or undefined.

$$\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

is sometimes another limit of type 0/0, that is,

$$\lim_{x\to c}f'(x)=0\quad\text{and}\quad\lim_{x\to c}g'(x)=0.$$

When this happens, l'Hospital's Rule can often be reapplied to $\lim_{x\to c}f'(x)/g'(x)$. The proof of l'Hospital's Rule in general is fairly long and uses the Mean Value Theorem. It will not be given here.

Here are some examples showing how the rule can be applied.

EXAMPLE 1 Find $\lim\limits_{x\to 1}\dfrac{(1/x)-1}{\sqrt{x}-1}$.

Both $(1/x) - 1$ and $\sqrt{x} - 1$ approach 0 as x approaches 1. The limit is thus of the form 0/0. Using l'Hospital's Rule,

$$\lim_{x \to 1} \frac{(1/x) - 1}{\sqrt{x} - 1} = \lim_{x \to 1} \frac{-x^{-2}}{\frac{1}{2}x^{-1/2}} = \frac{-1}{\frac{1}{2}} = -2.$$

EXAMPLE 2 Find $\lim_{x \to 0} \dfrac{\sqrt{x+1} - 1}{x^3}$.

The limit is of the form 0/0. The limit of $f'(x)/g'(x)$ as $x \to 0$ is ∞,

$$\lim_{x \to 0} \frac{d(\sqrt{x+1} - 1)/dx}{d(x^3)/dx} = \lim_{x \to 0} \frac{\frac{1}{2}(x+1)^{-1/2}}{3x^2} = \infty.$$

Thus by l'Hospital's Rule,

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x^3} = \infty.$$

EXAMPLE 3 Find $\lim_{x \to 3} \left(x + \dfrac{1}{x-3}\right)(\sqrt{x+1} - 2)$.

This limit is not in a form where we can apply l'Hospital's Rule. We must first use algebra to put it in another form,

$$\left(x + \frac{1}{x-3}\right)(\sqrt{x+1} - 2) = x(\sqrt{x+1} - 2) + \frac{\sqrt{x+1} - 2}{x-3}.$$

By elementary computations, $\lim_{x \to 3} x(\sqrt{x+1} - 2) = 3 \cdot 0 = 0$.

Using l'Hospital's Rule,

$$\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x-3} = \lim_{x \to 3} \frac{\frac{1}{2}(x+1)^{-1/2}}{1} = \frac{1}{2} \cdot 4^{-1/2} = \frac{1}{4}.$$

We then add the limits to get the desired answer,

$$\begin{aligned} \lim_{x \to 3} \left(x + \frac{1}{x-3}\right)(\sqrt{x+1} - 2) &= \lim_{x \to 3} x(\sqrt{x+1} - 2) + \lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x-3} \\ &= 0 + \tfrac{1}{4} = \tfrac{1}{4}. \end{aligned}$$

EXAMPLE 4 Find $\lim_{x \to 1} \dfrac{\dfrac{(x-3)}{4} + \dfrac{1}{x+1}}{(x-1)^2}$.

This limit is of the form 0/0. When l'Hospital's Rule is used the limit is still of the form 0/0. But when it is used a second time we can compute the limit.

$$\lim_{x \to 1} \frac{\dfrac{x-3}{4} + \dfrac{1}{x+1}}{(x-1)^2} = \lim_{x \to 1} \frac{\dfrac{1}{4} - \dfrac{1}{(x+1)^2}}{2(x-1)} = \lim_{x \to 1} \frac{2(x+1)^{-3}}{2} = \frac{1}{8}.$$

L'Hospital's Rule also holds true for other types of limits. That is, it holds true if $x \to c$ is everywhere replaced by one of the following.

$$x \to c^+, \qquad x \to c^-, \qquad x \to \infty, \qquad x \to -\infty.$$

EXAMPLE 5 Find $\displaystyle\lim_{x \to 0^+} \frac{\sqrt{x+4}-2}{\sqrt{x}}$.

The limit as $x \to 0$ does not exist because $\sqrt{x}$ is defined only for $x > 0$. However, the one-sided limit as $x \to 0^+$ has the form 0/0 and can be found by l'Hospital's Rule.

$$\lim_{x \to 0^+} \frac{\sqrt{x+4}-2}{\sqrt{x}} = \lim_{x \to 0^+} \frac{\frac{1}{2}(x+4)^{-1/2}}{\frac{1}{2}x^{-1/2}} = \lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{x+4}} = 0.$$

A second form of l'Hospital's Rule deals with the case where both $f(x)$ and $g(x)$ approach ∞ as x approaches c.

L'HOSPITAL'S RULE FOR ∞/∞

Suppose c is a real number, and in some deleted neighborhood of c, $f'(x)$ and $g'(x)$ exist and $g'(x) \neq 0$. Assume that

$$\lim_{x \to c} f(x) = \infty, \qquad \lim_{x \to c} g(x) = \infty.$$

If $\displaystyle\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists or is infinite, then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

The rule for ∞/∞ is exactly the same, word for word, as the rule for 0/0, except that 0 is replaced by ∞. We omit the proof, which is more difficult in the case ∞/∞. Actually, the assumption

$$\lim_{x \to c} f(x) = \infty$$

is not needed.

Again, l'Hospital's Rule for ∞/∞ also holds for the other types of limits,

$$x \to c^+, \qquad x \to c^-, \qquad x \to \infty, \qquad x \to -\infty.$$

EXAMPLE 6 Find $\displaystyle\lim_{x \to \infty} \frac{x+\sqrt{x+1}}{\sqrt{x}+\sqrt{x+1}}$.

By l'Hospital's Rule for ∞/∞,

$$\lim_{x \to \infty} \frac{x+\sqrt{x+1}}{\sqrt{x}+\sqrt{x+1}} = \lim_{x \to \infty} \frac{1+\dfrac{1}{2\sqrt{x}}}{\dfrac{1}{2\sqrt{x}}+\dfrac{1}{2\sqrt{x+1}}} = \infty.$$

Warning: Before using l'Hospital's Rule, check to see whether the limit is of the form 0/0 or ∞/∞. A common mistake is to use the rule when the limit is not of one of these forms.

EXAMPLE 7 Find $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-(1/x)}{x}$.

The limit has the form 0/1, so l'Hospital's Rule does not apply.

Correct: $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-(1/x)}{x}=\dfrac{\lim\limits_{x\to 1}(\sqrt{x}-(1/x))}{\lim\limits_{x\to 1}x}=\dfrac{0}{1}=0.$

Incorrect:

$$\lim_{x\to 1}\frac{\sqrt{x}-(1/x)}{x}=\lim_{x\to 1}\frac{d(\sqrt{x}-(1/x))/dx}{dx/dx}=\lim_{x\to 1}\left(\frac{1}{2\sqrt{x}}+\frac{1}{x^2}\right)=\frac{3}{2}.$$

PROBLEMS FOR SECTION 5.2

In Problems 1–34, evaluate the limit using l'Hospital's Rule.

1 $\lim\limits_{x\to 0}\dfrac{\sqrt{9+x}-3}{x}$

2 $\lim\limits_{t\to 1^+}\dfrac{1/t-1}{t^2-2t+1}$

3 $\lim\limits_{x\to 2}\dfrac{2-\sqrt{x+2}}{4-x^2}$

4 $\lim\limits_{t\to\infty}\dfrac{t+5-2t^{-1}-t^{-3}}{3t+12-t^{-2}}$

5 $\lim\limits_{y\to\infty}\dfrac{\sqrt{y+1}+\sqrt{y-1}}{y}$

6 $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$

7 $\lim\limits_{x\to 0}\dfrac{(1-x)^{1/4}-1}{x}$

8 $\lim\limits_{t\to 0}\left(t+\dfrac{1}{t}\right)((4-t)^{3/2}-8)$

9 $\lim\limits_{t\to 0^+}\left(\dfrac{1}{t}+\dfrac{1}{\sqrt{t}}\right)(\sqrt{t+1}-1)$

10 $\lim\limits_{x\to 0}\dfrac{x^2}{\sqrt{2x+1}-1}$

11 $\lim\limits_{u\to 1}\dfrac{(u-1)^3}{u^{-1}-u^2+3u-3}$

12 $\lim\limits_{t\to 0}\dfrac{2+1/x}{3-2/x}$

13 $\lim\limits_{u\to 0^+}\dfrac{1+5/\sqrt{u}}{2+1/\sqrt{u}}$

14 $\lim\limits_{u\to 0^+}\dfrac{3+u^{-1/2}+u^{-1}}{2+4u^{-1/2}}$

15 $\lim\limits_{x\to\infty}\dfrac{x+x^{1/2}+x^{1/3}}{x^{2/3}+x^{1/4}}$

16 $\lim\limits_{t\to\infty}\dfrac{1-\sqrt{t/(t+1)}}{2-\sqrt{(4t+1)/(t+2)}}$

17 $\lim\limits_{t\to\infty}\dfrac{1-t/(t-1)}{1-\sqrt{t/(t-1)}}$

18 $\lim\limits_{y\to-\infty}\dfrac{y+y^{-1}}{1+\sqrt{1-y}}$

19 $\lim\limits_{x\to 1}\dfrac{\int_1^x(1/t)\,dt}{\int_1^x 1/(2t+1)\,dt}$

20 $\lim\limits_{x\to\infty}\dfrac{\int_1^x\sqrt{t+t^{-1}}\,dt}{x\sqrt{x}}$

21 $\lim\limits_{x\to 0}\dfrac{\sin x}{x}$

22 $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$

23 $\lim\limits_{x\to 0}\dfrac{\sin(2x)}{x}$

24 $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x}$

25 $\lim\limits_{\theta\to\pi/2}\dfrac{\cos\theta}{\pi/2-\theta}$

26 $\lim\limits_{\theta\to\pi/2}\dfrac{\cos(3\theta)}{\pi/2-\theta}$

27 $\lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta}$

28 $\lim\limits_{\theta\to 0}\dfrac{\sin(2\theta)}{\sin(5\theta)}$

29 $\lim\limits_{t\to 0}\dfrac{e^t-1}{t}$

30 $\lim\limits_{t\to 0}\dfrac{t^2}{e^t-t-1}$

31 $\lim_{t\to 1}\dfrac{\ln t}{t-1}$

32 $\lim_{t\to 0}\dfrac{\ln(t^2+1)}{t}$

33 $\lim_{x\to 1}\dfrac{x\ln x}{x^2-1}$

34 $\lim_{x\to 0}\dfrac{\sin(2x)}{\ln(x+1)}$

In Problems 35–52, evaluate the limit by l'Hospital's Rule or otherwise.

35 $\lim_{x\to 1}\dfrac{x^{1/4}-1}{x}$

36 $\lim_{x\to 1^+}\dfrac{\sqrt{x}}{x-1}$

37 $\lim_{x\to 1^+}\dfrac{\sqrt{x}-1}{x-1}$

38 $\lim_{x\to\infty}\dfrac{x^{-1}+x^{-1/2}}{x+x^{-1/2}}$

39 $\lim_{x\to\infty}\dfrac{x+x^{-2}}{2x+x^{-2}}$

40 $\lim_{x\to\infty}\dfrac{5+x^{-1}}{1+2x^{-1}}$

41 $\lim_{x\to\infty}\dfrac{4x}{\sqrt{2x^2+1}}$

42 $\lim_{x\to 0}\dfrac{3x^2+x+2}{x-4}$

43 $\lim_{x\to 0}\dfrac{\sqrt{x+1}-1}{\sqrt{x+4}-2}$

44 $\lim_{x\to 0}\dfrac{\sqrt{x+1}-1}{\sqrt{x+2}-1}$

45 $\lim_{x\to 0^+}\dfrac{\sqrt{x+1}+1}{\sqrt{x+1}-1}$

46 $\lim_{x\to 0}\dfrac{\sqrt{x^2+1}-1}{\sqrt{x+1}-1}$

47 $\lim_{x\to\infty}(x+5)\left(\dfrac{1}{2x}+\dfrac{1}{x+2}\right)$

48 $\lim_{x\to 0^+}(x+5)\left(\dfrac{1}{2x}+\dfrac{1}{x+2}\right)$

49 $\lim_{x\to 1}(x+5)\left(\dfrac{1}{2x}+\dfrac{1}{x+2}\right)$

50 $\lim_{x\to 2}\dfrac{x^3-6x-2}{x^3+4}$

51 $\lim_{x\to 2}\dfrac{x^3-6x-2}{x^3-4x}$

52 $\lim_{x\to 1^+}\dfrac{x^3+4x+8}{2x^3-2}$

☐ 53 Suppose f and g are continuous in a neighborhood of a and $g(a)\neq 0$. Show that

$$\lim_{x\to a}\frac{\int_a^x f(t)\,dt}{\int_a^x g(t)\,dt}=\frac{f(a)}{g(a)}.$$

5.3 LIMITS AND CURVE SKETCHING

By definition, $\lim_{x\to c} f(x)=L$ means that for every *hyperreal* number x which is infinitely close but not equal to c, $f(x)$ is infinitely close to L. What does $\lim_{x\to c} f(x)=L$ tell us about $f(x)$ for *real* numbers x? It turns out that if $\lim_{x\to c} f(x)=L$, then for every real number x which is close to but not equal to c, $f(x)$ is close to L.

In the next section we shall justify the above intuitive statement by a mathematical theorem. The main difficulty is to make the word "close" precise. For the time being we shall simply illustrate the idea with some examples.

EXAMPLE 1 Consider the limit $\lim_{x\to 0}\dfrac{2/x+1}{1/x-1}=2.$

This limit is evaluated by letting $x\neq 0$ be infinitesimal:

$$\frac{2/x+1}{1/x-1}=\frac{2+x}{1-x},$$

$$\lim_{x\to 0}\frac{2/x+1}{1/x-1}=st\left(\frac{2+x}{1-x}\right)=\frac{st(2+x)}{st(1-x)}=\frac{2+0}{1-0}=2.$$

Let us see what happens if instead of taking x to be infinitely small we take x to be a "small" real number. We shall make a table of values of

$$f(x) = \frac{2/x + 1}{1/x - 1}$$

for various small x.

x	$f(x) = \dfrac{2/x + 1}{1/x - 1}$	$f(x)$ to four places
0.1	21/9	2.3333
0.01	201/99	2.0303
0.001	2001/999	2.0030
0.0001	20001/9999	2.0003
−0.1	19/11	1.7364
−0.01	199/101	1.9703
−0.001	1999/1001	1.9970
−0.0001	19999/10001	1.9997

We see that as x gets closer and closer to zero, $f(x)$ gets closer and closer to 2. With a calculator, the student should try this for some of the limits on pages 124 and 241.

The table helps us to draw the graph of the curve $y = f(x)$. Although the point $(0, 2)$ is not on the graph, we know that when x is close to 0, $f(x)$ is close to 2, and draw the graph accordingly. The graph is drawn in Figure 5.3.1.

Other types of limits also give information which is useful in drawing graphs. For instance, if $\lim_{x\to c} f(x) = \infty$, then for every number x which is close to but not equal to c, the value of $f(x)$ is large. And if $\lim_{x\to\infty} f(x) = L$, then for every large real number x, $f(x)$ is close to L.

In both the above statements, if we replace "close" by "infinitely close" and "large" by "infinitely large" we get our official definition of a limit. We give two more examples.

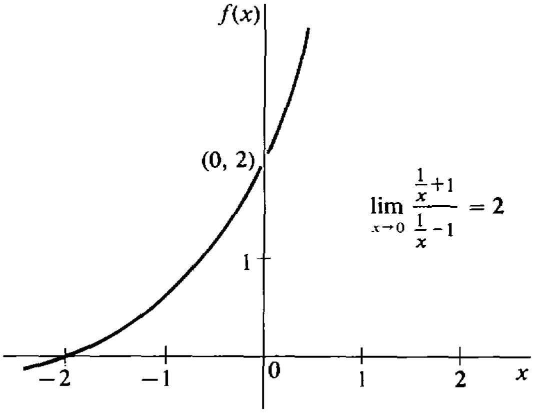


Figure 5.3.1

EXAMPLE 2 Consider the limit $\displaystyle\lim_{x\to 2} \frac{1}{(x-2)^2} = \infty.$

For x infinitely close but not equal to 2, $1/(x-2)^2$ is positive infinite. Let us make a table of values when x is a real number close to but not equal to 2.

x	$f(x)$
2.1	100
2.01	10000
2.001	1000000
1.9	100
1.99	10000
1.999	1000000

As x gets closer and closer to 2, $f(x)$ gets larger and larger.

EXAMPLE 3 $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{(x-2)^2}\right)=1.$

For infinitely large x, $1+1/(x-2)^2$ is infinitely close to 1. Here is a table of values of $1+1/(x-2)^2$ for large real x.

x	$1+\dfrac{1}{(x-2)^2}$
12	1.01
102	1.0001
1002	1.000001
10002	1.00000001

As x gets large, $1+1/(x-2)^2$ gets close to 1. Also notice that

$$\lim_{x\to-\infty}\left(1+\frac{1}{(x-2)^2}\right)=1,$$

and for large negative x, $1+1/(x-2)^2$ is close to 1.

In Chapter 3 we showed how to use the first and second derivatives to sketch the graph of a function which is continuous on a closed interval. In the next example we shall sketch the graph of the function $f(x)=1+1/(x-2)^2$. But this time the function is discontinuous at $x=2$, and the domain is the whole real line except for the point $x=2$. Our method uses not only the values but also the limits of the function and its first derivative.

EXAMPLE 4 Sketch the curve $f(x)=1+\dfrac{1}{(x-2)^2}.$

The first two derivatives are

$$f'(x)=-2(x-2)^{-3}\qquad f''(x)=6(x-2)^{-4}.$$

The first and second derivatives are never zero. $f(x)$ is undefined at $x=2$. In our table we shall show the values of $f(x)$ and its first two derivatives at a point on each side of $x = 2$. We shall also show the limits of $f(x)$ and its first derivative as $x \to -\infty$, $x \to 2^-$, $x \to 2^+$, and $x \to \infty$. (We will not need the limits of $f''(x)$.)

	$f(x)$	$f'(x)$	$f''(x)$	Comments
$\lim_{x\to-\infty}$	1	0		horizontal
$x = 1$	2	2	6	increasing, $\cup$
$\lim_{x\to 2^-}$	∞	∞		vertical
$\lim_{x\to 2^+}$	∞	$-\infty$		vertical
$x = 3$	2	-2	6	decreasing, $\cup$
$\lim_{x\to\infty}$	1	0		horizontal

The first line of the table, $\lim_{x\to-\infty}$, shows that for large negative x the curve is close to 1 and its slope is nearly horizontal. The second line, $x = 1$, shows that the curve is increasing and concave upward in the interval $(-\infty, 2)$, and passes through the point $(1, 2)$ with a slope of 2. The third line, $\lim_{x\to 2^-}$, shows that just before $x = 2$ the curve is far above the x-axis and its slope is nearly vertical. Going through the table in this way, we are able to sketch the curve as in Figure 5.3.2.

The curve approaches the dotted horizontal line $y = 1$ and the dotted vertical line $x = 2$. These lines are called *asymptotes* of the curve.

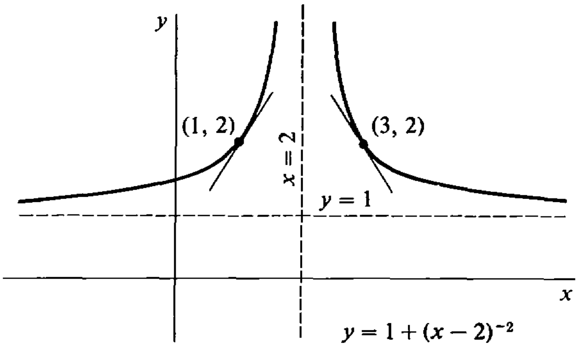


Figure 5.3.2

Suppose the function f and its derivative f' exist and are continuous at all but a finite number of points of an interval I. The following procedure can be used in sketching the curve $y = f(x)$.

Step 1 First carry out the procedure outlined in Section 3.9 concerning the first and second derivative.

Step 2 Compute $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.
(They may either be real numbers, $+\infty$, $-\infty$, or may not exist.)

Step 3 At each point c of I where f is discontinuous, compute $f(c)$, $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$.
(Some or all of these quantities may be undefined.)

Step 4 Compute $\lim_{x\to\infty} f'(x)$ and $\lim_{x\to-\infty} f'(x)$.

Step 5 At each point where f' is discontinuous, compute $f(c)$, $\lim_{x\to c^-} f'(x)$ and $\lim_{x\to c^-} f'(x)$.

We shall now work several more examples; the steps in computing the limits are left to the student.

EXAMPLE 5 $f(x) = x^{3/5}$.

Then $f'(x) = \frac{3}{5}x^{-2/5}$, $f''(x) = -\frac{6}{25}x^{-7/5}$.

At the point $x = 0$, $f(x) = 0$ and $f'(x)$ does not exist. We first plot a few points, compute the necessary limits, and make a table.

	$f(x)$	$f'(x)$	$f''(x)$	Comments
$\lim_{x\to-\infty}$	$-\infty$	0		horizontal
$x = -1$	-1	3/5	6/25	increasing, ∪
$\lim_{x\to 0^-}$	0	∞		vertical
$x = 0$	0	undef.		
$\lim_{x\to 0^+}$	0	∞		vertical
$x = 1$	1	3/5	$-6/25$	increasing, ∩
$\lim_{x\to\infty}$	∞	0		horizontal

Figure 5.3.3 is a sketch of the curve.

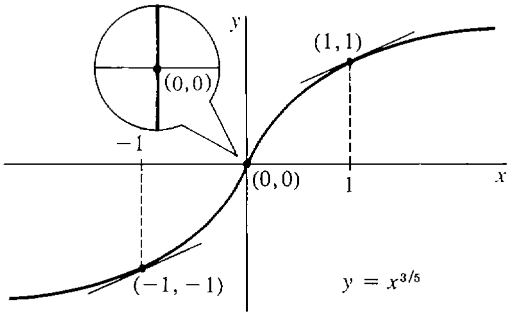


Figure 5.3.3

The behavior as x approaches $-\infty$, ∞, and zero are described by the limits we have computed. As x approaches either $-\infty$ or ∞, $f(x)$ gets large but the slope becomes more nearly horizontal. As x approaches zero the curve becomes nearly vertical, increasing from left to right, so we have a vertical tangent line at $x = 0$.

EXAMPLE 6 $f(x) = x^{4/5}$.

Then $\qquad f'(x) = \frac{4}{5}x^{-1/5}, \qquad f''(x) = -\frac{4}{25}x^{-6/5}$.

$f'(x)$ is undefined at $x = 0$. We make the table:

	$f(x)$	$f'(x)$	$f''(x)$	Comments
$\lim_{x\to-\infty}$	∞	0		horizontal
$x = -1$	1	$-4/5$	$-4/25$	decreasing, $\frown$
$\lim_{x\to0^-}$	0	$-\infty$		vertical
$x = 0$	0	undef.		
$\lim_{x\to0^+}$	0	∞		vertical
$x = 1$	1	$4/5$	$-4/25$	increasing, $\frown$
$\lim_{x\to\infty}$	∞	0		horizontal

With this information we can sketch the curve in Figure 5.3.4.

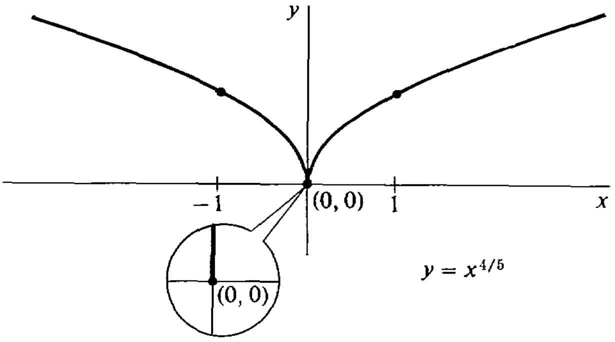


Figure 5.3.4

This time the limits of the derivative as x approaches zero show that there is a cusp at $x = 0$, with the curve decreasing when $x < 0$ and increasing when $x > 0$.

EXAMPLE 7 Sketch the curve $f(x) = \dfrac{\cos x}{\sin x}$ for $0 < x < 2\pi$.

$f(x)$ and $f'(x)$ are undefined at $x = \pi$ because the denominator $\sin \pi$ is zero. The first two derivatives are

$$f'(x) = -\frac{1}{\sin^2 x}, \qquad f''(x) = 2\frac{\cos x}{\sin^3 x}.$$

Thus $f'(x)$ is always negative, and $f''(x) = 0$ when $x = \pi/2$, $3\pi/2$. Here is the table:

	$f(x)$	$f'(x)$	$f''(x)$	Comments
$\lim_{x\to 0^+}$	∞	$-\infty$		vertical
$\pi/4$	1	$-1/2$	$+$	decreasing, ∪
$\pi/2$	0	-1	0	decreasing, inflection
$3\pi/4$	-1	$-1/2$	$-$	decreasing, ∩
$\lim_{x\to \pi^-}$	$-\infty$	$-\infty$		vertical
$\lim_{x\to \pi^+}$	∞	$-\infty$		vertical
$5\pi/4$	1	$-1/2$	$+$	decreasing, ∪
$3\pi/2$	0	-1	0	decreasing, inflection
$7\pi/4$	-1	$-1/2$	$-$	decreasing, ∩
$\lim_{x\to 2\pi^-}$	$-\infty$	$-\infty$		vertical

Notice that the table from π to 2π is just a repeat of the table from 0 to π. This is because

$$\frac{\cos(x+\pi)}{\sin(x+\pi)} = \frac{-\cos x}{-\sin x} = \frac{\cos x}{\sin x}.$$

The curve is sketched in Figure 5.3.5.

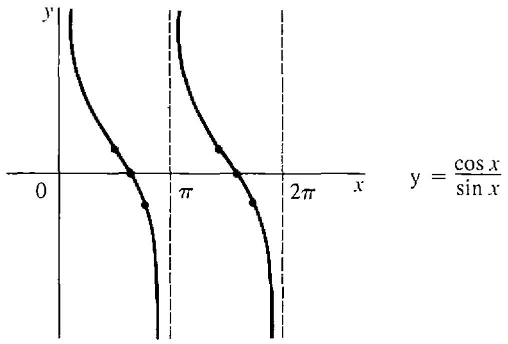


Figure 5.3.5

PROBLEMS FOR SECTION 5.3

1 This figure is a sketch of a curve $y = f(x)$. At which points $x = c$ do the following happen?

(a) f is discontinuous at c

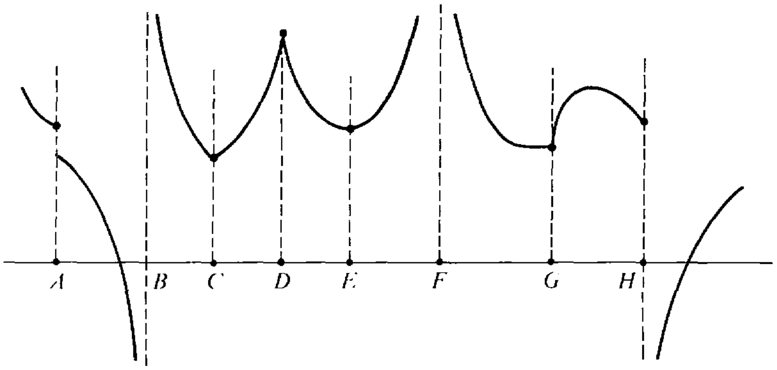
(b) $\lim_{x\to c^+} f(x)$ does not exist

(c) $\lim_{x\to c^-} f(x)$ does not exist

(d) f is not differentiable at c

(e) $\lim_{x\to c^+} f'(x)$ does not exist

(f) $\lim_{x\to c^-} f'(x)$ does not exist.



In Problems 2–42, sketch the graph of $f(x)$. Use a table of values of $f(x), f'(x), f''(x)$, and limits of $f(x)$ and $f'(x)$. Then check your answer by using a graphics calculator to draw the graph.

2 $f(x) = 2 - \frac{1}{2}x^2$

3 $f(x) = x^2 - 2x$

4 $f(x) = x^3 - x$

5 $f(x) = x^2 - \frac{1}{6}x^3$

6 $f(x) = \frac{1}{2}x^4 - x^2$

7 $f(x) = x^3 - \frac{1}{4}x^4$

8 $f(x) = 1 + \dfrac{1}{x}$

9 $f(x) = \dfrac{1}{2 - x}$

10 $f(x) = x^2 + \dfrac{2}{x}$

11 $f(x) = \dfrac{1}{x^2}$

12 $f(x) = x^2 + \dfrac{1}{x^2}$

13 $f(x) = \sqrt{x}$

14 $f(x) = \sqrt{2 - x}$

15 $f(x) = \dfrac{1}{\sqrt{x}}$

16 $f(x) = 1 - \dfrac{1}{\sqrt{x}}$

17 $f(x) = \sqrt[3]{x}$

18 $f(x) = 2 - (x - 1)^{1/3}$

19 $f(x) = \dfrac{x - 1}{x + 1}$

20 $f(x) = \dfrac{2x}{1 - x}$

21 $f(x) = \dfrac{1}{x^2 + 1}$

22 $f(x) = \dfrac{x}{x^2 + 1}$

23 $f(x) = \dfrac{x^2}{x^2 + 1}$

24 $f(x) = \dfrac{1}{x^2 - 1}$

25 $f(x) = \dfrac{x}{x^2 - 1}$

26 $f(x) = \dfrac{x^2}{x^2 - 1}$

27 $f(x) = x^{2/3}$

28 $f(x) = 2 + (x - 1)^{2/3}$

29 $f(x) = \sqrt{4 - x^2}$

30 $f(x) = 4\sqrt{1 - x^2}$

31 $f(x) = 1 - \sqrt{1 - x^2}$

32 $f(x) = \sqrt{x^2 - 1}$

33 $y = \dfrac{1}{\sin x}, \quad 0 < x < 2\pi$

34 $y = \dfrac{1}{\cos x}, \quad 0 < x < 2\pi$

35 $y = \tan x, \quad 0 \le x \le 2\pi$

36 $y = \tan^2 x, \quad -\pi \le x \le \pi$

37 $y = \dfrac{1}{\sin x + \cos x}, \quad 0 \le x \le 2\pi$

38 $y = \dfrac{1}{\sin x \cos x}, \quad 0 \le x \le 2\pi$

39 $f(x) = -\sqrt{x^2 - 4}$

40 $f(x) = 2 - \sqrt{x^2 + 4}$

41 $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$

42 $f(x) = \dfrac{1}{\sqrt{1 - x^2}}$

In Problems 43–55, graph the given function.

43 $f(x) = |x| - 1$

44 $f(x) = 1 - |2x|$

45 $f(x) = |2x - 1|$

46 $f(x) = 2 + \left|\dfrac{x}{2} - 3\right|$

47 $f(x) = 2x + |x - 2|$

48 $f(x) = x^2 + |x|$

49 $f(x) = x^2 + |x + 1|$

50 $f(x) = |x^2 - 1|$

51 $f(x) = \sqrt{|x|}$

52 $f(x) = x/|x|$

53 $f(x) = x + \dfrac{x}{|x|}$

54 $f(x) = \dfrac{x^3 - x}{|x|}$

☐ **55** $f(x) = x\sqrt{1 + 1/x^2}$

5.4 PARABOLAS

In this section we shall study the graph of the equation

$$y = ax^2 + bx + c,$$

which is a U-shaped curve called a vertical parabola. We begin with the general definition of a parabola in the plane.

Recall that the distance between a point P and a line L is the length of the perpendicular line from P to L, as in Figure 5.4.1. If we are given a line L and a point F not on L, the set of all points equidistant from L and F will form a U-shaped curve that passes midway between L and F. This curve is a parabola, shown in Figure 5.4.2.

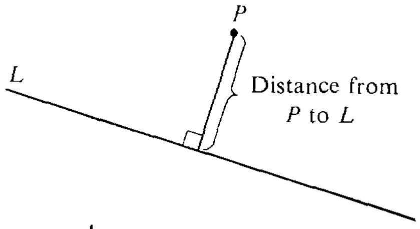


Figure 5.4.1

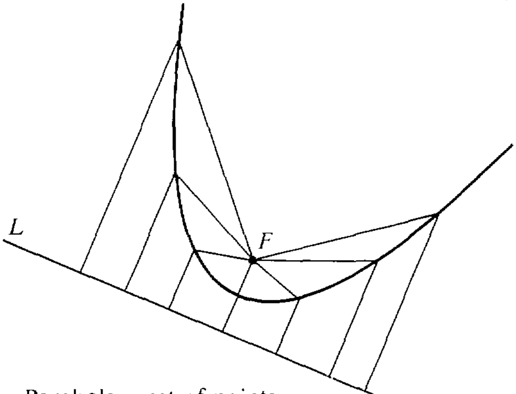


Figure 5.4.2 Parabola = set of points equidistant from L and F.

DEFINITION OF PARABOLA

Given a line L and a point F not on the line, the set of all points equidistant from L and F is called the ***parabola*** *with* ***directrix*** *L and* ***focus*** *F.*

The line through the focus perpendicular to the directrix is called the *axis* of the parabola. The point where the parabola crosses the axis is called the *vertex*. These are illustrated in Figure 5.4.3.

As we can see from the figure, the parabola is symmetric about its axis. That is, if we fold the page along the axis, the parabola will fold upon itself. The vertex is just the point halfway between the focus and directrix. It is the point on the parabola which is closest to the directrix and focus.

When a ball is thrown into the air, its path is the parabola shown in Figure 5.4.4, with the highest point at the vertex.

Telescope mirrors and radar antennae are in the shape of parabolas. This is done because all light rays coming from the direction of the axis will be reflected to a single point, the focus (see Figure 5.4.5). For the same reason, reflectors for searchlights and automobile headlights are shaped like parabolas, with the light at the focus.

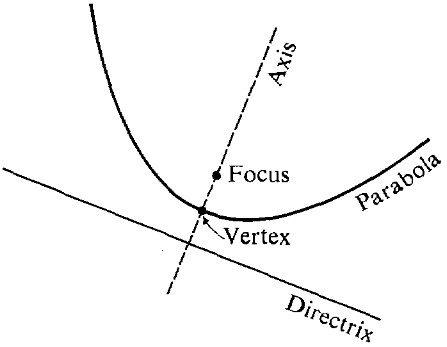


Figure 5.4.3

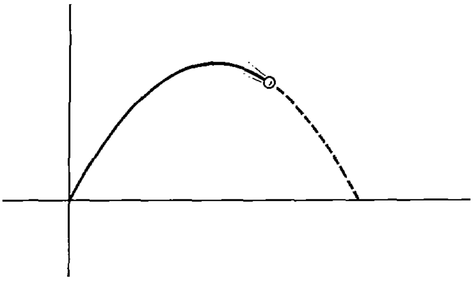


Figure 5.4.4

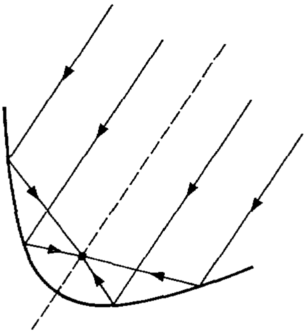


Figure 5.4.5

A parabola with a vertical axis (and horizontal directrix) is called a *vertical* parabola. The vertex of a vertical parabola is either the highest or lowest point, because it is the point closest to the directrix.

EXAMPLE 1 Find an equation for the vertical parabola with directrix $y = -1$ and focus $F(0, 1)$ (Figure 5.4.6).

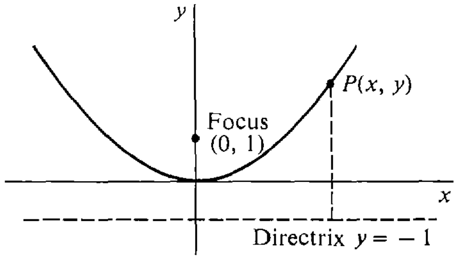


Figure 5.4.6

Given a point $P(x, y)$, the perpendicular from P to the directrix is a vertical line of length $\sqrt{(y+1)^2}$. Thus

$$\text{distance from } P \text{ to directrix} = \sqrt{(y+1)^2}.$$

Also,
$$\text{distance from } P \text{ to focus} = \sqrt{x^2 + (y-1)^2}.$$

The point P lies on the parabola exactly when these distances are equal,

$$\sqrt{(y+1)^2} = \sqrt{x^2 + (y-1)^2}.$$

The equation of a parabola is particularly simple if the coordinate axes are chosen so that the vertex is at the origin and the focus is on the y-axis. The parabola will then be vertical and have an equation of the form $y = ax^2$.

THEOREM 1

The graph of the equation

$$y = ax^2$$

(where $a \neq 0$) is the parabola with focus $F(0, 1/4a)$ and directrix $y = -1/(4a)$. Its vertex is $(0, 0)$, and its axis is the y-axis.

PROOF Let us find the equation of the parabola with focus $F(0, d)$ and directrix $y = -d$, shown in Figure 5.4.7.

Our plan is to show that the equation is $y = ax^2$ where $d=1/(4a)$. Given a point $P(x, y)$, the perpendicular from P to the directrix is a vertical line of length $\sqrt{(y+d)^2}$. Thus

$$\text{distance from } P \text{ to directrix} = \sqrt{(y+d)^2}.$$

Also,
$$\text{distance from } P \text{ to focus} = \sqrt{x^2 + (y-d)^2}.$$

The point P lies on the parabola exactly when these distances are equal,

$$\sqrt{(y+d)^2} = \sqrt{x^2 + (y-d)^2}.$$

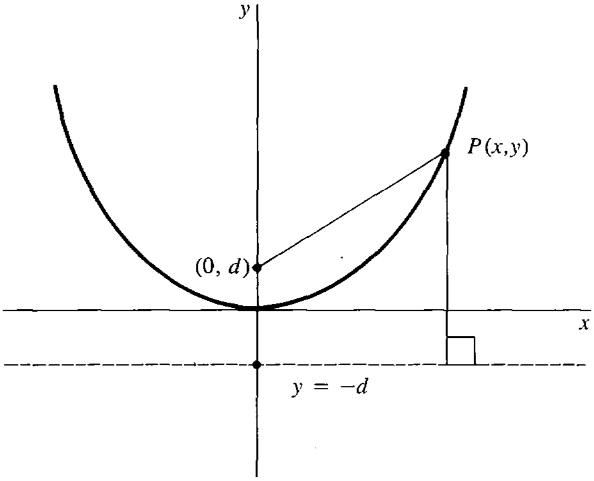


Figure 5.4.7

Simplifying we get

$$\begin{aligned}(y+d)^2 &= x^2+(y-d)^2\\ y^2+2yd+d^2 &= x^2+y^2-2yd+d^2\\ 4yd &= x^2\\ y &= \frac{1}{4d}x^2.\end{aligned}$$

Putting $a=1/4d$, we have $d=1/4a$ where $y = ax^2$ is the equation of the parabola.

Note that if a is negative, the focus will be below the x-axis and the directrix above the x-axis.

EXAMPLE 2 Find the focus and directrix of the parabola

$$y = -(1/2)\,x^2.$$

In Theorem 1, $a = -1/2$ and $d=1/4a = -\frac{1}{2}$. The focus is $F(0, -\frac{1}{2})$, and the directrix is $y = \frac{1}{2}$.

The next theorem shows that the graph of $y = ax^2 + bx + c$ is exactly like the graph of $y = ax^2$, except that its vertex is at the point (x_0, y_0) where the curve has slope zero. The focus and directrix are still at a distance $1/(4a)$ above and below the vertex.

THEOREM 2

The graph of the equation

$$y = ax^2 + bx + c$$

(where $a \neq 0$) is a vertical parabola. Its vertex is at the point (x_0, y_0) where

the curve has slope zero, the focus is $F(x_0, y_0 + 1/4a)$, *and the directrix is* $y = y_0 - 1/4a$.

PROOF We first compute x_0. The curve $y = ax^2 + bx + c$ has slope $dy/dx = 2ax + b$. The slope is zero when $2ax + b = 0$, $x = -b/2a$. Thus

$$x_0 = -b/2a.$$

Let p be the parabola with focus $F(x_0, y_0 + 1/4a)$ and directrix $y = y_0 - 1/4a$. Put $X = x - x_0$ and $Y = y - y_0$. In terms of X and Y, the focus and directrix are at

$$(X, Y) = (0, 1/4a), \qquad Y = -1/4a.$$

By Theorem 1, p has the equation

$$Y = aX^2,$$

or

$$y - y_0 = a(x - x_0)^2,$$

$$y = ax^2 - 2ax_0x + (ax_0^2 + y_0).$$

Substituting $-b/2a$ for x_0, we have

$$y = ax^2 + bx + (b^2/4a + y_0).$$

This shows that the parabola p and the curve $y = ax^2 + bx + c$ differ at most by a constant. Moreover, the point (x_0, y_0) lies on the curve. (x_0, y_0) is also the vertex of the parabola p, where $(X, Y) = (0, 0)$. Therefore the curve and the parabola are the same.

EXAMPLE 3 Find the vertex, focus and directrix of the parabola

$$y = 2x^2 - 5x + 4.$$

First find the point x_0 where the slope is 0.

$$\frac{dy}{dx} = 4x - 5.$$

Then

$$4x_0 - 5 = 0,$$

$$x_0 = \tfrac{5}{4}.$$

Substitute to find y_0.

$$y_0 = 2(x_0)^2 - 5x_0 + 4 = \tfrac{7}{8}.$$

The vertex is

$$(x_0, y_0) = (\tfrac{5}{4}, \tfrac{7}{8}).$$

We have $a = 2$, so $1/4a = \frac{1}{8}$. By Theorem 2, the focus is

$$\left(x_0, y_0 + \frac{1}{4a}\right) = \left(\frac{5}{4}, 1\right).$$

The directrix is

$$y = y_0 - \frac{1}{4a}, \qquad y = \frac{3}{4}.$$

The vertex, axis, focus, and directrix can be used to sketch quickly the graph of a vertical parabola.

GRAPHING A PARABOLA $y = ax^2 + bx + c$

Step 1 Make a table of values of x, y, dy/dx, and d^2y/dx^2 at $x \to -\infty$, $x = -b/2a$ (the vertex), and $x \to \infty$.

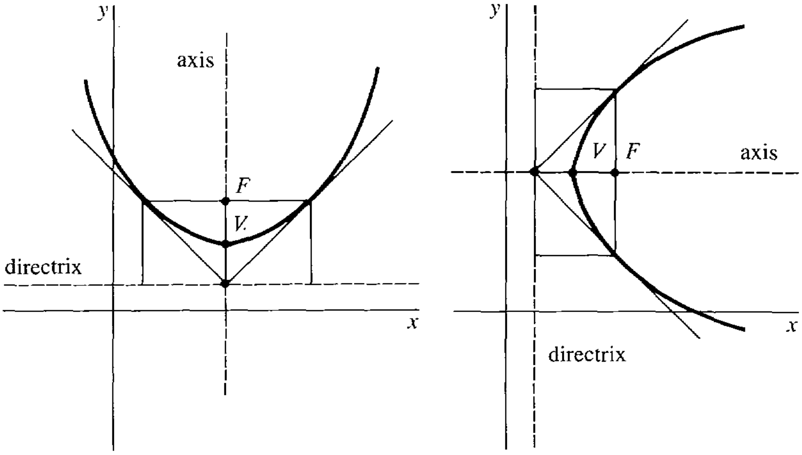
Step 2 Compute the axis, vertex, focus, and directrix, and draw them.

Step 3 Draw the two squares with sides along the axis and directrix and a corner at the focus. The two new corners level with the focus, P and Q, are on the parabola because they are equidistant from the focus and the directrix.

Step 4 Draw the diagonals of the squares through P and Q. These are the tangent lines to the parabola at P and Q. (The proof of this fact is left as a problem.)

Step 5 Draw the parabola through the vertex, P, and Q, using the table and tangent lines. The parabola should be symmetrical about the axis $x = -b/2a$. See Figure 5.4.8(a).

A *horizontal parabola* $x = ay^2 + by + c$ can be graphed by the same method with the roles of x and y interchanged, as in Figure 5.4.8(b).



(a) Vertical (b) Horizontal

Figure 5.4.8

EXAMPLE 2 (Continued) Sketch the parabola $y = -\frac{1}{2}x^2$.

The first two derivatives are

$$\frac{dy}{dx} = -x, \qquad \frac{d^2y}{dx^2} = -1.$$

The only critical point is at $x = 0$. The table of values follows.

x	y	dy/dx	d^2y/dx^2	Comments
$\lim_{x \to -\infty}$	$-\infty$	∞		vertical
$x = 0$	0	0	-1	max, ∩
$\lim_{x \to \infty}$	$-\infty$	$-\infty$		vertical

The parabola is drawn in Figure 5.4.9, using Steps 1–5.

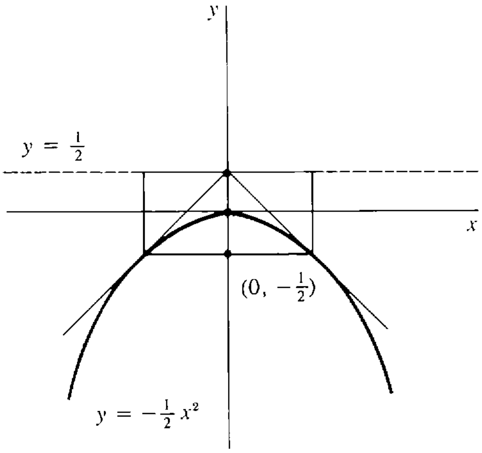


Figure 5.4.9

EXAMPLE 3 (Continued) Sketch the parabola $y = 2x^2 - 5x + 4$.

The first two derivatives are

$$\frac{dy}{dx} = 4x - 5, \qquad \frac{d^2y}{dx^2} = 4.$$

The only critical point is at the vertex, where $x = \frac{5}{4}$. The table of values follows.

x	y	dy/dx	d^2y/dx^2	Comments
$\lim_{x \to -\infty}$	∞	$-\infty$		vertical
5/4	7/8	0	+	min, ∪
$\lim_{x \to \infty}$	∞	∞		vertical

The parabola is drawn in Figure 5.4.10, again using Steps 1–5.

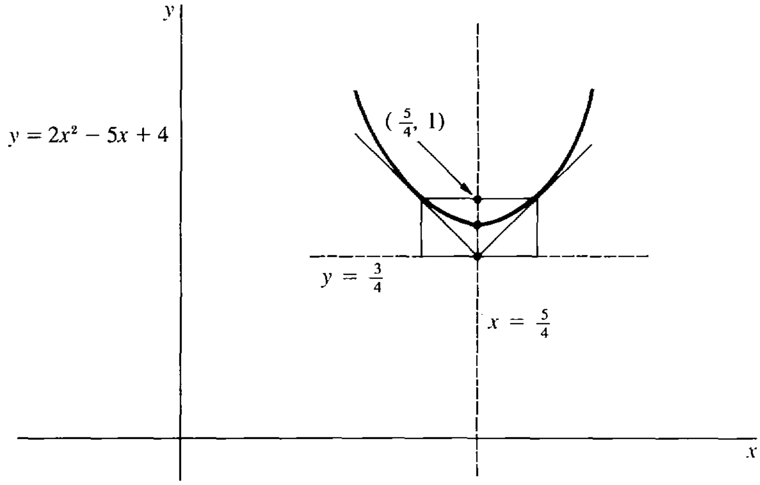


Figure 5.4.10

We can now sketch the graph of any equation of the form

$$Ax^2 + Dx + Ey + F = 0.$$

In the ordinary case where both A and E are different from zero, proceed as follows. First, solve the equation for y, obtaining the new equation

$$y = -\frac{A}{E}x^2 - \frac{D}{E}x - \frac{F}{E}.$$

Second, use the method in this section to sketch the graph, which will be a vertical parabola. There are also two *degenerate cases*. If $A = 0$, the graph is a straight line. If $E = 0$, then y does not appear at all, and the graph is either two vertical lines, one vertical line, or empty.

We can also sketch the graph of any equation of the form

$$Cy^2 + Dx + Ey + F = 0.$$

In the ordinary case where C and D are different from zero, the graph will be a horizontal parabola.

PROBLEMS FOR SECTION 5.4

In Problems 1–14, find the focus and directrix, and sketch the given parabola.

1 $y = 2x^2$
2 $y = \frac{1}{2}x^2$
3 $y = -x^2$
4 $y = 2 - x^2$
5 $y = x^2 - 2x$
6 $y = x^2 + 2x + 1$
7 $y = 2x^2 + x - 2$
8 $y = x^2 - x + 1$
9 $y = 3 + x - x^2$
10 $y = 1 - x - x^2$
11 $y = \frac{1}{2}x^2 + x - 1$
12 $y = \frac{1}{3}x^2 - x$
13 $y = (x - 2)^2$
14 $y = 2(x + 1)^2$
15 $x = y^2$
16 $x = 2y^2 - 4$
17 $x = -y^2 + y + 1$
18 $x = 3 - (y - 2)^2$

19 Find the equation of the parabola with directrix $y = 0$ and focus $F(2, 2)$.

20 Find the equation of the parabola with directrix $y = -1$ and focus $F(0, 0)$.

21 Find the focus of the parabola with directrix $y = 1$ and vertex $(1, 2)$.

22 Find the equation of the parabola with focus $(-1, -1)$ and vertex $(-1, 0)$.

5.5 ELLIPSES AND HYPERBOLAS

In this section we shall study two important types of curves, the ellipses and hyperbolas. The intersection of a circular cone and a plane will always be either a parabola, an ellipse, a hyperbola, or one of three degenerate cases—one line, two lines, or a point. For this reason, parabolas, ellipses, and hyperbolas are called *conic sections*. We begin with the definition of an ellipse in the plane.

DEFINITION OF ELLIPSE

*Given two points, F_1 and F_2, and a constant, L, the **ellipse** with foci F_1 and F_2 and length L is the set of all points the sum of whose distances from F_1 and F_2 is equal to L.*

If the two foci F_1 and F_2 are the same, the ellipse is just the circle with center at the focus and diameter L. Circles are discussed in Section 1.1.

We shall concentrate on the case where the foci F_1 and F_2 are different. The ellipse will be an oval curve shown in Figure 5.5.1. The orbit of a planet is an ellipse with the sun at one focus. The eye sees a tilted circle as an ellipse.

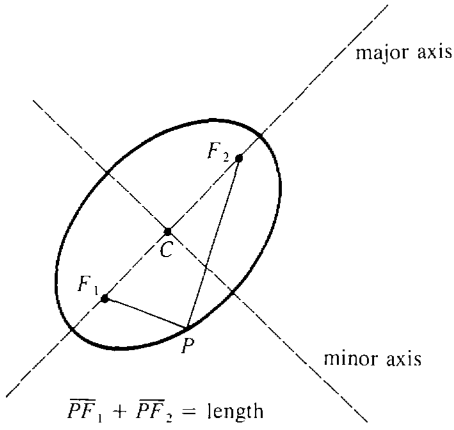


Figure 5.5.1 Ellipse

The line through the foci F_1 and F_2 is called the *major axis* of the ellipse. The point on the major axis halfway between the foci is called the *center*. The line through the center perpendicular to the major axis is called the *minor axis*.

An ellipse is symmetric about both its major and its minor axes. That is, for any point P on the ellipse, the mirror image of P on the other side of either axis is also on the ellipse. The equation of an ellipse has a simple form when the major and minor axes are chosen for the x-axis and y-axis.

THEOREM 1

For any positive a and b, the graph of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is an ellipse with its center at the origin. There are three cases:

(i) $a = b$. *The ellipse is a circle of radius a.*

(ii) $a > b$. *This is a horizontal ellipse, whose major axis is the x-axis, and whose minor axis is the y-axis. The length is 2a. The foci are at $(-c, 0)$ and $(c, 0)$, where c is found by*

$$c^2 = a^2 - b^2.$$

(iii) $a < b$. *This is a vertical ellipse whose major axis is the y-axis and whose minor axis is the x-axis. The length is 2b. The foci are at $(0, -c)$ and $(0, c)$, where c is found by*

$$c^2 = b^2 - a^2.$$

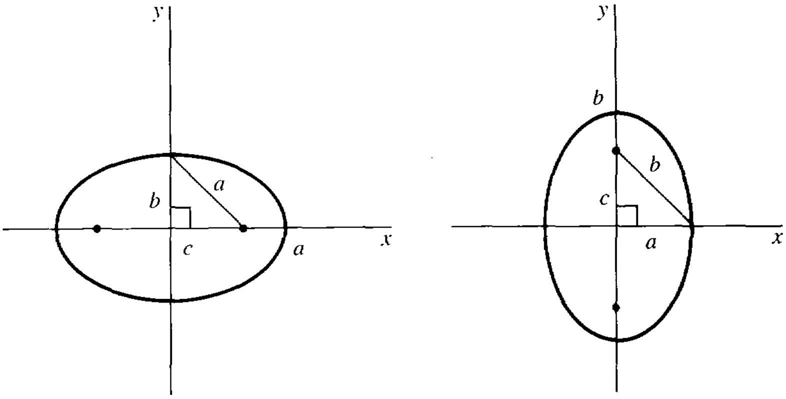


Figure 5.5.2 Horizontal $c^2 = a^2 - b^2$ Vertical $c^2 = b^2 - a^2$

This theorem is illustrated by Figure 5.5.2. Here is the proof in case (ii), $a > b$. A point $P(x, y)$ is on the ellipse with foci $(-c, 0)$, $(c, 0)$ and length $2a$ if and only if the sum of the distances from P to the foci is $2a$. That is,

$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

Rewrite this as

$$\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}.$$

Square both sides:

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2.$$

Simplify:

$$a\sqrt{(x + c)^2 + y^2} = a^2 + cx.$$

Square both sides again:

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2.$$

Collect the x^2 and y^2 terms and simplify.

$$x^2(a^2 - c^2) + y^2(a^2) = a^4 - a^2c^2 = a^2(a^2 - c^2).$$

Using the equation $b^2 = a^2 - c^2$, write this as

$$x^2b^2 + y^2a^2 = a^2b^2.$$

Finally, divide by a^2b^2 to obtain the required equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Setting $x = 0$ we see that the ellipse meets the y-axis at the two points $y = \pm b$. Also, it meets the x-axis at $x = \pm a$. Since all terms are ≥ 0, at every point on the ellipse we have

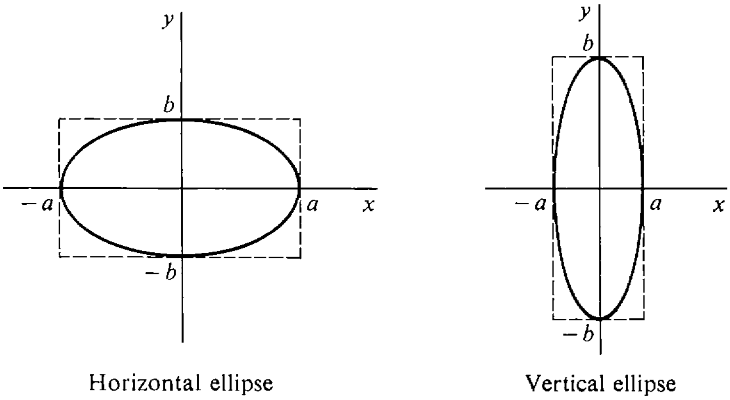
$$\frac{x^2}{a^2} \leq 1, \qquad -a \leq x \leq a$$

and

$$\frac{y^2}{b^2} \leq 1, \qquad -b \leq y \leq b.$$

Using these facts we can easily sketch the ellipse. It is an oval curve inscribed in the rectangle bounded by the lines $x = \pm a$, $y = \pm b$.

Figure 5.5.3 shows a *horizontal* ellipse (where $a > b$) and a *vertical* ellipse (where $a < b$).



Horizontal ellipse Vertical ellipse

Figure 5.5.3

EXAMPLE 1 Sketch the curve $\dfrac{x^2}{9} + y^2 = 1.$

The curve is an ellipse that cuts the x-axis at ± 3 and the y-axis at ± 1. To sketch the curve, we first draw the rectangle $x = \pm 3$, $y = \pm 1$ with dotted lines and then inscribe the ellipse in the rectangle. The ellipse, shown in Figure 5.5.4, is horizontal.

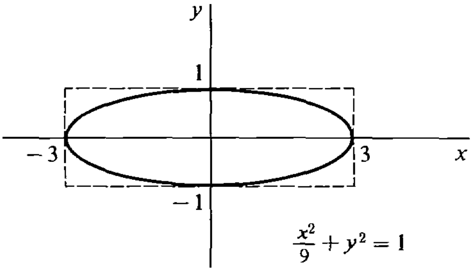


Figure 5.5.4

EXAMPLE 2 Sketch the curve $4x^2 + y^2 = 9$ and find the foci.

The equation may be rewritten as

$$\tfrac{4}{9}x^2 + \tfrac{1}{9}y^2 = 1.$$

The graph (Figure 5.5.5) is a vertical ellipse cutting the x-axis at $\pm\frac{3}{2}$ and the y-axis at ± 3.

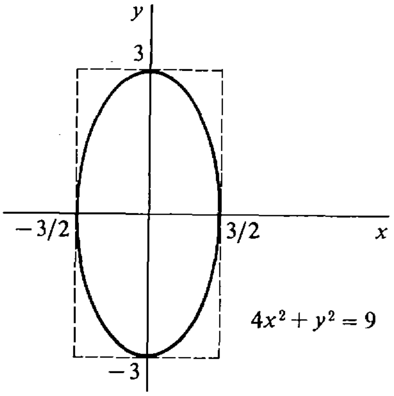


Figure 5.5.5

By Theorem 1, the foci are on the y-axis at $(0, \pm c)$. We compute c from the equation

$$c^2 = b^2 - a^2.$$

a and b are the x and y intercepts of the ellipse, $a = \frac{3}{2}$, $b = 3$. Thus

$$c^2 = 3^2 - (\tfrac{3}{2})^2 = \tfrac{27}{4}$$
$$c = \sqrt{\tfrac{27}{2}} \sim 2.598.$$

The foci are at $(0, \pm 2.598)$.

We turn next to the hyperbola. A hyperbola, like an ellipse, has two foci. However, the distances between the foci and a point on the hyperbola must have a constant difference instead of a constant sum.

DEFINITION OF HYPERBOLA

Given two distinct points, F_1 and F_2, and a constant, l, the ***hyperbola*** *with foci F_1 and F_2 and difference l is the set of all points the difference of whose distances from F_1 and F_2 is equal to l.*

In this definition, l must be a positive number less than the distance between the foci. A hyperbola will have two separate branches, each shaped like a rounded V. On one branch the points are closer to F_1 than F_2; and on the other branch they are closer to F_2 than F_1. Figure 5.5.6 shows a typical hyperbola. The path of a comet on an orbit that will escape the solar system is a hyperbola with the sun at one focus. The shadow of a cylindrical lampshade on a wall is a hyperbola (the section of the light cone cut by the wall).

The line through the foci is the *transverse axis* of the hyperbola, and the point on the axis midway between the foci is the *center*. The hyperbola crosses the transverse axis at two points called the *vertices*. The line through the center perpendicular to the transverse axis is the *conjugate axis*. The hyperbola never crosses its conjugate axis. A hyperbola is symmetric about both axes. A simple equation is obtained when the transverse and conjugate axes are chosen for the coordinate axes.

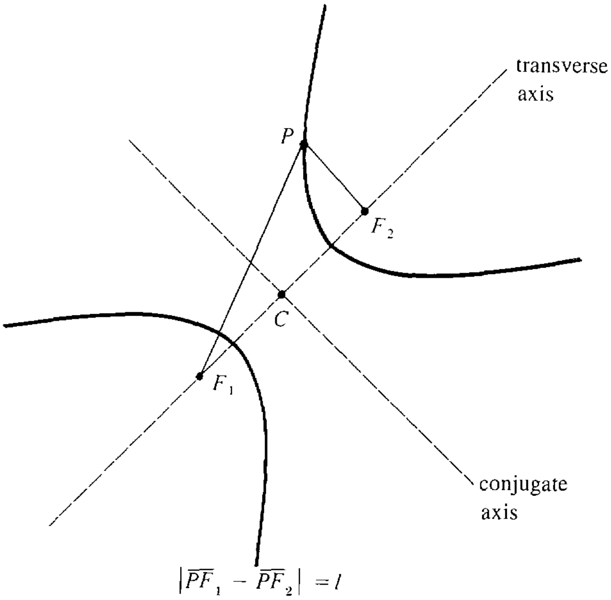


Figure 5.5.6 Hyperbola

THEOREM 2

For any positive a and b, the graph of the equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

is a hyperbola with its center at the origin. Its transverse axis is the y-axis, *and its conjugate axis is the x-axis. The vertices are at* $(0, \pm b)$, *and the foci are at* $(0, \pm c)$, *where c is found by*

$$a^2 + b^2 = c^2.$$

The graph of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is a hyperbola with similar properties with the roles of x, a and y, b reversed. The proof of Theorem 2 uses a computation like the proof of Theorem 1 on ellipses and is omitted.

Using derivatives and limits, we can get additional information that is helpful in sketching the graph of a hyperbola. By solving the equation

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

for y as a function of x, we see that the upper and lower branches have the equations

$$\text{upper branch:} \qquad y = \frac{b}{a}\sqrt{a^2 + x^2},$$

$$\text{lower branch:} \qquad y = -\frac{b}{a}\sqrt{a^2 + x^2}.$$

We concentrate on the upper branch. Its first two derivatives, after some algebraic simplification, come out to be

$$\frac{dy}{dx} = \frac{bx}{a\sqrt{a^2 + x^2}}, \qquad \frac{d^2y}{dx^2} = ab(a^2 + x^2)^{-3/2}.$$

Thus the first derivative is zero only at $x = 0$ (the vertex), and the second derivative is always positive. We have the following table of values for the upper branch.

x	y	dy/dx	d^2y/dx^2	Comments
$\lim_{x \to -\infty}$	∞	$-b/a$	0	decreasing
0	b	0	b/a^2	minimum, ∪
$\lim_{x \to \infty}$	∞	b/a	0	increasing

All the limit computations are easy except for dy/dx, which we work out for $x \to \infty$. Let H be positive infinite.

$$\begin{aligned}\lim_{x \to \infty} dy/dx &= \lim_{x \to \infty} \frac{bx}{a\sqrt{a^2 + x^2}} \\ &= st\left[\frac{bH}{a\sqrt{a^2 + H^2}}\right] \\ &= st\left[\frac{b}{a\sqrt{a^2H^{-2} + 1}}\right] = \frac{b}{a}.\end{aligned}$$

We carry out a similar computation for the limit as $x \to -\infty$.

$$\begin{aligned}\lim_{x\to-\infty}\frac{dy}{dx} &= \lim_{x\to-\infty}\frac{bx}{a\sqrt{a^2+x^2}}\\ &= st\left[\frac{b(-H)}{a\sqrt{a^2+(-H)^2}}\right]\\ &= st\left[\frac{-b}{a\sqrt{a^2H^{-2}+1}}\right] = -\frac{b}{a}.\end{aligned}$$

The table shows that the upper branch is almost a straight line with slope $-b/a$ for large negative x and almost a straight line with slope b/a for large positive x. In fact, we shall show now that the lines

$$y = bx/a, \qquad y = -bx/a$$

are asymptotes of the hyperbola. That is, as x approaches ∞ or $-\infty$, the distance between the line and the hyperbola approaches zero. We show that the upper branch approaches the line $y = bx/a$ as $x \to \infty$; that is,

$$\lim_{x\to\infty}\left[\frac{b}{a}\sqrt{a^2+x^2} - \frac{bx}{a}\right] = 0.$$

Let H be positive infinite. Then

$$\begin{aligned}\frac{b}{a}\sqrt{a^2+H^2} - \frac{bH}{a} &= \frac{b}{a}[\sqrt{a^2+H^2} - H]\\ &= \frac{b}{a}\left[\frac{(\sqrt{a^2+H^2}-H)(\sqrt{a^2+H^2}+H)}{\sqrt{a^2+H^2}+H}\right]\\ &= \frac{b}{a}\frac{a^2+H^2-H^2}{\sqrt{a^2+H^2}+H}\\ &= ab(\sqrt{a^2+H^2}+H)^{-1}.\end{aligned}$$

This is infinitesimal, so the limit is zero. Here are the steps for graphing a hyperbola $y^2/b^2 - x^2/a^2 = 1$.

GRAPHING A HYPERBOLA $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

Step 1 Compute the values of a and b from the equation. Draw the rectangle with sides $x = \pm a$, $y = \pm b$.

Step 2 Draw the diagonals of the rectangle. They will be the asymptotes.

Step 3 Mark the vertices of the hyperbola at the points $(0, \pm b)$.

Step 4 Draw the upper and lower branches of the hyperbola. The upper branch has a minimum at the vertex $(0, b)$, is concave upward, and approaches the diagonal asymptotes from above. The lower branch is a mirror image. See Figure 5.5.7.

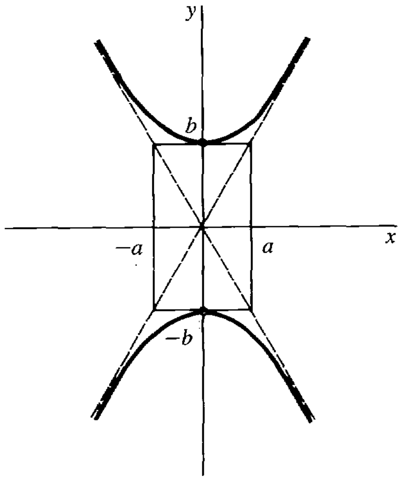


Figure 5.5.7

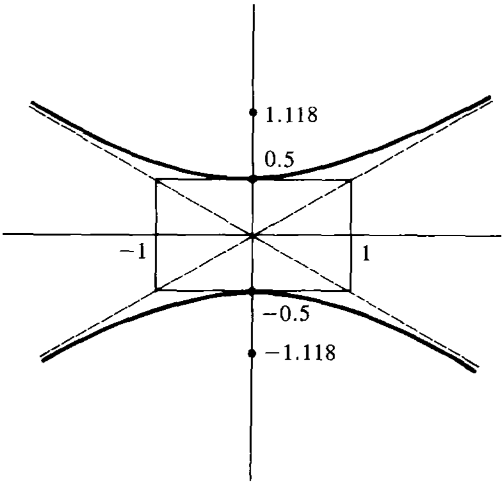


Figure 5.5.8

A hyperbola of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is graphed in a similar manner, but with the roles of x and y reversed. There is a left branch and a right branch, which are vertical at the vertices $(\pm a, 0)$.

EXAMPLE 3 Sketch the hyperbola $4y^2 - x^2 = 1$ and find its foci.

First compute a and b.

$$4y^2 = y^2/b^2, \qquad b = \tfrac{1}{2}$$
$$x^2 = x^2/a^2, \qquad a = 1.$$

The rectangle has sides $x = \pm 1$, $y = \pm\frac{1}{2}$, and the vertices are at $(0, \pm\frac{1}{2})$. The hyperbola is sketched using Steps 1–4 in Figure 5.5.8. The foci are at $(0, \pm c)$ where

$$c^2 = a^2 + b^2 = 1^2 + (\tfrac{1}{2})^2 = 1.25$$
$$c = \sqrt{1.25} \sim 1.118.$$

Using the method of this section, we can sketch the graph of any equation of the form

$$Ax^2 + Cy^2 + F = 0.$$

In the ordinary case where A, C, and F are all different from zero, rewrite the equation as

$$A_1x^2 + C_1y^2 = 1,$$

where $A_1 = -A/F$, $C_1 = -C/F$. There are four cases depending on the signs of A_1 and C_1, which are listed in Table 5.5.1.

Table 5.5.1

A_1	C_1	Graph of $A_1x^2 + C_1y^2 = 1$	
>0	>0	ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
>0	<0	hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
<0	>0	hyperbola	$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$
<0	<0	empty	

If one or two of A, C, and F are zero, the graph will be degenerate (two lines, one line, a point, or empty).

PROBLEMS FOR SECTION 5.5

In Problems 1–12, find the foci and sketch the given ellipse or hyperbola.

1 $x^2 + 4y^2 = 1$

2 $x^2 + \frac{1}{4}y^2 = 1$

3 $\frac{1}{4}x^2 + 4y^2 = 1$

4 $\frac{1}{25}x^2 + \frac{1}{9}y^2 = 1$

5 $9x^2 + 4y^2 = 16$

6 $x^2 + 9y^2 = 4$

7 $y^2 - 4x^2 = 1$

8 $y^2 - x^2 = 4$

9 $9y^2 - x^2 = 4$

10 $4y^2 - 4x^2 = 1$

11 $x^2 - y^2 = 1$

12 $\frac{x^2}{9} - \frac{y^2}{4} = 1$

13 Prove that the hyperbola $x^2/a^2 - y^2/b^2 = 1$ has the two asymptotes $y = bx/a$ and $y = -bx/a$.

5.6 SECOND DEGREE CURVES

A *second degree equation* is an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \tag{1}$$

The graph of such an equation will be a conic section: a parabola, ellipse, hyperbola, or one of several degenerate cases. In Section 5.4 we saw that the graph of a second degree equation of one of the forms

$$Ax^2 + Dx + Ey + F = 0 \tag{2}$$

or

$$Cy^2 + Dx + Ey + F = 0 \tag{3}$$

is a parabola or degenerate. In Section 5.5 we saw that the graph of a second degree equation of the form

$$(4) \qquad Ax^2 + Cy^2 + F = 0$$

is an ellipse, a hyperbola, or degenerate.

In this and the next section we shall see how to describe and sketch the graph of any second degree equation. We will begin with the Discriminant Test, which shows at once whether a nondegenerate curve is a parabola, ellipse, or hyperbola. The next topic in this section will be translation of axes, which can change any second degree equation with no xy-term,

$$(5) \qquad Ax^2 + Cy^2 + Dx + Ey + F = 0,$$

into an equation of one of the simple forms (2), (3), or (4).

In the following section we will study rotation of axes, which can change any second degree equation into an equation of the form (5) with no xy-term. We will then be able to deal with any second degree equation by using first rotation and then translation of axes.

Here is the Discriminant Test.

DEFINITION

The quantity $B^2 - 4AC$ is called the ***discriminant*** *of the equation*

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

DISCRIMINANT TEST

If we ignore the degenerate cases, the graph of a second degree equation is:

A parabola if the discriminant is zero.
An ellipse if the discriminant is negative.
A hyperbola if the discriminant is positive.

For example, the equation

$$xy - 1 = 0$$

has positive discriminant $1^2 - 4 \cdot 0 = 1$, and its graph is a hyperbola. The equation

$$2x^2 + xy + y^2 - 1 = 0$$

has negative discriminant $1^2 - 4 \cdot 2 \cdot 1 = -7$, and its graph is an ellipse.

The degenerate graphs that can arise are: two straight lines, one straight line, one point, and the empty graph. The Discriminant Test alone does not tell whether or not the graph is degenerate. However, a degenerate case can usually be recognized when one tries to sketch the graph. For the remainder of this section we shall ignore the degenerate cases.

We now turn to the method of Translation of Axes. This method is useful for graphing a second degree equation with no xy-term,

$$Ax^2 + Cy^2 + Dx + Ey + F = 0.$$

If A or C is zero, the graph will be a horizontal or vertical parabola, which can be graphed by the method of Section 5.4. If both A and C are nonzero, the graph turns out to be an ellipse or hyperbola with horizontal and vertical axes X and Y, as in

Figure 5.6.1. In the method of Translation of Axes, we take X and Y as a new pair of coordinate axes and get a new equation for the curve in the simple form

$$AX^2 + CY^2 + F_1 = 0.$$

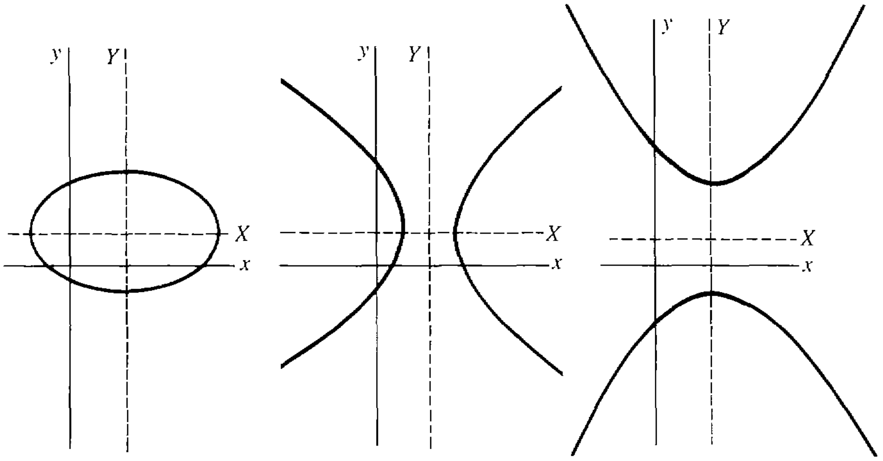


Figure 5.6.1

This curve can be sketched as in Section 5.5. The name "Translation of Axes" means that the original coordinate axes x and y are replaced by new coordinate axes X and Y, which are parallel to the original axes.

The new axes are found using a procedure from algebra called "completing the squares." This procedure changes an expression like $Ax^2 + Dx$ into a perfect square plus a constant.

FORMULA FOR COMPLETING THE SQUARES

Let A be different from zero. Then

$$Ax^2 + Dx = AX^2 + K,$$

where

$$X = x + \frac{D}{2A}, \qquad K = \frac{-D^2}{4A}.$$

For example,

$$4x^2 - 3x = 4X^2 - 9/16$$

where $X = x - \frac{3}{8}$.

We shall illustrate the method of Translation of Axes with an example and then describe the method in general.

EXAMPLE 1 Sketch the curve $4x^2 - y^2 - 16x - 2y + 11 = 0$.

Step 1 Apply the Discriminant Test to determine the type of curve.

$$B^2 - 4AC = 0^2 - 4 \cdot 4 \cdot (-1) = 16.$$

The discriminant is positive, so the graph is a hyperbola.

Step 2 Simplify by completing the squares. This is done by putting

$$X = x + \frac{D}{2A}, \qquad Y = y + \frac{E}{2C}$$

and writing the original equation in terms of X and Y.

$$X = x + \frac{-16}{2 \cdot 4} = x - 2, \qquad x = X + 2$$

$$Y = y + \frac{-2}{2 \cdot (-1)} = y + 1, \qquad y = Y - 1$$

$$4(X+2)^2 - (Y-1)^2 - 16(X+2) - 2(Y-1) + 11 = 0$$

$$4(X^2 + 4X + 4) - 16(X+2) - (Y^2 - 2Y + 1) - 2(Y-1) + 11 = 0.$$

The X and Y terms cancel, and

$$4X^2 + 16 - 32 - Y^2 - 1 + 2 + 11 = 0,$$

$$4X^2 - Y^2 - 4 = 0.$$

Step 3 Draw dotted lines for the X and Y axes, and sketch the curve as in Section 5.5. This is a hyperbola in the (X, Y)-plane. The X-axis is the line $Y = 0$, or $y = -1$. The Y-axis is the line $X = 0$, or $x = 2$. The graph is shown in Figure 5.6.2.

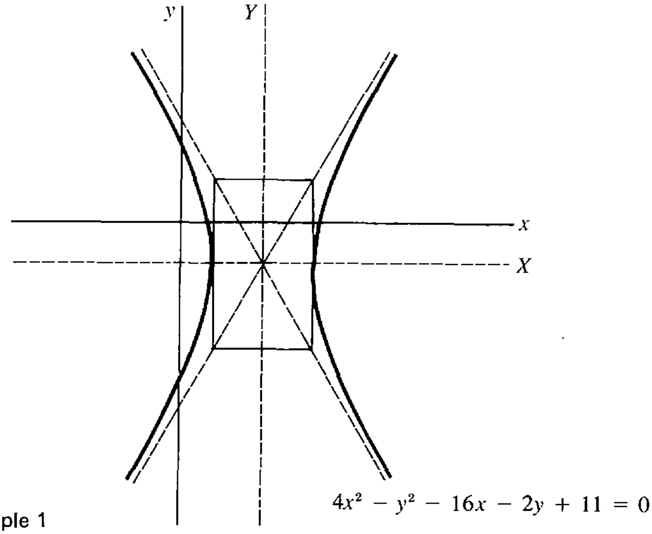


Figure 5.6.2 Example 1

METHOD OF TRANSLATION OF AXES

When to Use *To graph an equation of the form* $Ax^2 + Cy^2 + Dx + Ey + F = 0$ *where* A *and* C *are both nonzero.*

Step 1 *Use the Discriminant Test to determine the type of curve.*

Step 2 *Completing the Squares: Put*

$$X = x + \frac{D}{2A}, \qquad Y = y + \frac{E}{2C}$$

and rewrite the original equation in terms of X and Y. The new equation will have the simple form

$$Ax^2 + Cy^2 + F_1 = 0,$$

where F_1 is a new constant.

Step 3 *Draw dotted lines for the X and Y axes and sketch the curve as in Section* 5.5.

PROBLEMS FOR SECTION 5.6

In Problems 1–6, given that the graph is nondegenerate, use the Discriminant Test to determine whether the graph is a parabola, ellipse, or hyperbola.

1 $x^2 + 2xy - 3y^2 + 5x + 6y - 100 = 0$

2 $4x^2 - 8xy + 6y^2 + 10x - 2y - 20 = 0$

3 $4x^2 + 4xy + y^2 + 7x + 8y = 0$

4 $9x^2 + 6xy + y^2 + 6x - 22 = 0$

5 $x^2 + 5xy + 10y^2 - 16 = 0$

6 $4xy + 5x - 10y + 1 = 0$

In Problems 7–18, use the method of Translation of Axes to sketch the curve.

7 $x^2 + y^2 - 4x + 3 = 0$

8 $x^2 + y^2 + 2x - 6y + 6 = 0$

9 $x^2 - y^2 + 4x - 2y + 2 = 0$

10 $-x^2 + y^2 + 8x - 6y - 16 = 0$

11 $x^2 + 4y^2 - 4x + 24y + 36 = 0$

12 $4x^2 - 9y^2 + 8x + 18y - 41 = 0$

13 $9x^2 - 4y^2 - 36x - 24y - 36 = 0$

14 $-x^2 + 4y^2 + 16y + 12 = 0$

15 $-x^2 + 3y^2 + 8x + 30y + 56 = 0$

16 $5x^2 + 2y^2 + 10x + 12y + 28 = 0$

17 $16x^2 + 9y^2 - 320x - 108y + 1780 = 0$

18 $25x^2 + 4y^2 + 250x - 40y + 625 = 0$

5.7 ROTATION OF AXES

We have seen how to graph any second degree equation with no xy-term. These graphs are parabolas, ellipses, or hyperbolas with vertical and horizontal axes. When the equation has a nonzero xy-term, the graph will have diagonal axes. By rotating the axes, one can get new coordinate axes in the proper direction. The method will give us a new equation that has no xy-term and can be graphed by our previous method.

Suppose the x and y axes are rotated counterclockwise by an angle α, and the new coordinate axes are called X and Y, as in Figure 5.7.1. A point P in the plane will have a pair of coordinates (x, y) in the old coordinate system and (X, Y) in the new coordinate system. The old and new coordinates of P are related to each other by the equations for rotation of axes.

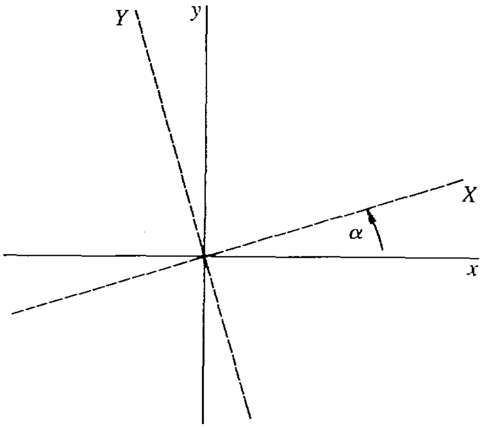


Figure 5.7.1 Rotation of Axes

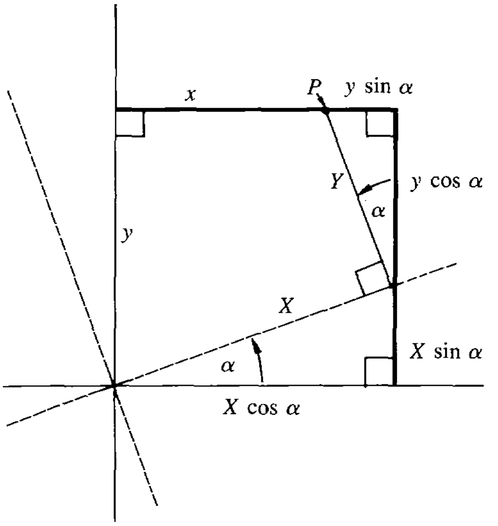


Figure 5.7.2

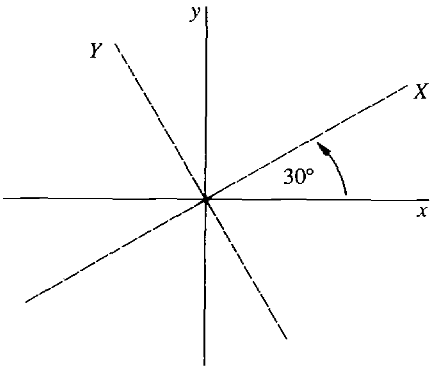


Figure 5.7.3

EQUATIONS FOR ROTATION OF AXES

$$x = X\cos\alpha - Y\sin\alpha, \qquad y = X\sin\alpha + Y\cos\alpha.$$

These equations can be seen directly from Figure 5.7.2. If we substitute the equations for rotation of axes into a second degree equation in x and y, we get a new second degree equation in the coordinates X and Y.

EXAMPLE 1 Find the equation of the curve

$$xy - 4 = 0,$$

with respect to the new coordinate axes X and Y formed by a counterclockwise rotation of 30 degrees (Figure 5.7.3).

In this example,

$$\alpha = 30^\circ, \qquad \sin\alpha = \frac{1}{2}, \qquad \cos\alpha = \frac{\sqrt{3}}{2}.$$

Thus $$x = -\frac{\sqrt{3}}{2}X - \frac{1}{2}Y, \qquad y = \frac{1}{2}X + \frac{\sqrt{3}}{2}Y.$$

Substitute into the original equation and collect terms.

$$xy - 4 = 0,$$

$$\left(\frac{\sqrt{3}}{2}X - \frac{1}{2}Y\right)\cdot\left(\frac{1}{2}X + \frac{\sqrt{3}}{2}Y\right) - 4 = 0,$$

$$\frac{\sqrt{3}}{4}X^2 + \frac{1}{2}XY - \frac{\sqrt{3}}{4}Y^2 - 4 = 0.$$

Given any second degree equation

(1) $$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

and any angle of rotation α, one can substitute the equations of rotation and collect terms to get a new second degree equation in the X and Y coordinates,

(2) $$A_1X^2 + B_1XY + C_1Y^2 + D_1X + E_1Y + F_1 = 0.$$

It can be shown that the discriminant is unchanged by the rotation; that is,

$$B^2 - 4AC = B_1^2 - 4A_1C_1.$$

This gives a useful check on the computations.

In Example 1 above, the original discriminant is

$$B^2 - 4AC = 1^2 - 4\cdot 0\cdot 0 = 1.$$

The new equation has the same discriminant,

$$B_1^2 - 4A_1C_1 = \left(\frac{1}{2}\right)^2 - 4\left(\frac{\sqrt{3}}{4}\right)\left(-\frac{\sqrt{3}}{4}\right) = \frac{1}{4} + \frac{3}{4} = 1.$$

The trouble with Example 1 is that the new equation is more complicated than the original equation, and in particular there is still a nonzero XY-term. We would like to be able to choose the angle of rotation α so that the new equation has no XY-term, because we could then sketch the curve. The next theorem tells us which angle of rotation is needed.

THEOREM 1

Given a second degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

with B nonzero. Rotate the coordinate axes counterclockwise through an angle α for which

$$\cot(2\alpha) = \frac{A - C}{B}.$$

Then the equation

$$A_1X^2 + B_1XY + C_1Y^2 + D_1X + E_1Y + F_1 = 0$$

with respect to the new coordinate axes X and Y has XY-term $B_1 = 0$.

This theorem can be proved as follows. When the rotation equations are substituted and terms collected, the XY coefficient B_1 comes out to be

$$B_1 = B(\cos^2\alpha - \sin^2\alpha) - 2(A - C)\sin\alpha\cos\alpha.$$

From trigonometry,

$$\cos^2\alpha - \sin^2\alpha = \cos(2\alpha), \qquad 2\sin\alpha\cos\alpha = \sin(2\alpha).$$

Thus
$$B_1 = B\cos(2\alpha) - (A - C)\sin(2\alpha).$$

So $B_1 = 0$ if and only if

$$B\cos(2\alpha) - (A - C)\sin(2\alpha) = 0,$$

$$\frac{\cos(2\alpha)}{\sin(2\alpha)} - \frac{A - C}{B} = 0,$$

or
$$\cot(2\alpha) = \frac{A - C}{B}.$$

As shown in Figure 5.7.4, α is the angle between the original coordinate axes and the axes of the parabola, ellipse, or hyperbola.

We are now ready to use rotation of axes to sketch a second degree curve. We illustrate the method for the curve introduced in Example 1.

EXAMPLE 2 Sketch the curve $xy - 4 = 0$.

Step 1 Apply the Discriminant Test to find the type of curve.

$$B^2 - 4AC = 1^2 - 4 \cdot 0 \cdot 0 = 1.$$

The discriminant is positive, so the curve is a hyperbola.

Step 2 Find an angle α with

$$\cot(2\alpha) = \frac{A - C}{B}.$$

$$\cot(2\alpha) = \frac{0 - 0}{1} = 0.$$

$$2\alpha = 90^\circ, \qquad \alpha = 45^\circ.$$

Step 3 Change coordinate axes using the rotation equations.

$$\cos\alpha = \frac{\sqrt{2}}{2}, \qquad \sin\alpha = \frac{\sqrt{2}}{2}.$$

$$x = X\cos\alpha - Y\sin\alpha = \frac{\sqrt{2}}{2}X - \frac{\sqrt{2}}{2}Y.$$

$$y = X\sin\alpha + Y\cos\alpha = \frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}Y.$$

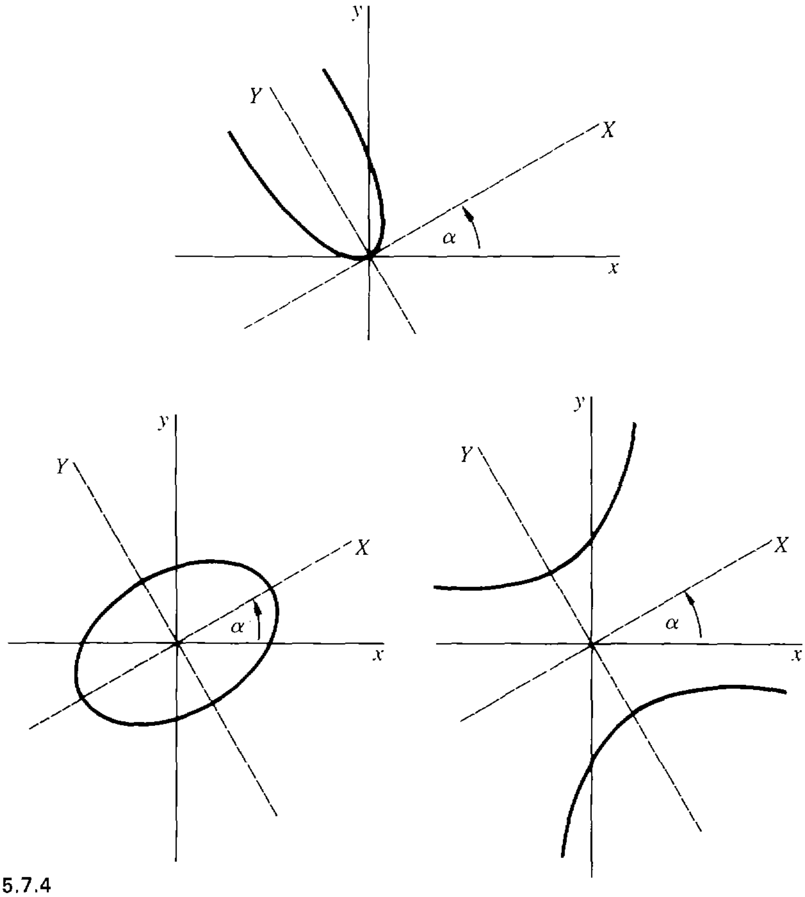


Figure 5.7.4

Substituting, we get

$$xy - 4 = 0,$$

$$\left(\frac{\sqrt{2}}{2}X - \frac{\sqrt{2}}{2}Y\right) \cdot \left(\frac{\sqrt{2}}{2}X + \frac{\sqrt{2}}{2}Y\right) - 4 = 0,$$

$$\frac{1}{2}X^2 - \frac{1}{2}Y^2 - 4 = 0.$$

As a check, the discriminant is still $0^2 - 4 \cdot (\frac{1}{2}) \cdot (-\frac{1}{2}) = 1$.

Step 4 Draw the X and Y axes as dotted lines and sketch the curve.

The new axes are found by rotating the old axes by $\alpha = 45°$. The curve is shown in Figure 5.7.5.

METHOD OF ROTATION OF AXES

When to Use *To graph an equation of the form* $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ *where* B *is nonzero.*

Step 1 *Use the Discriminant Test to determine the type of curve.*

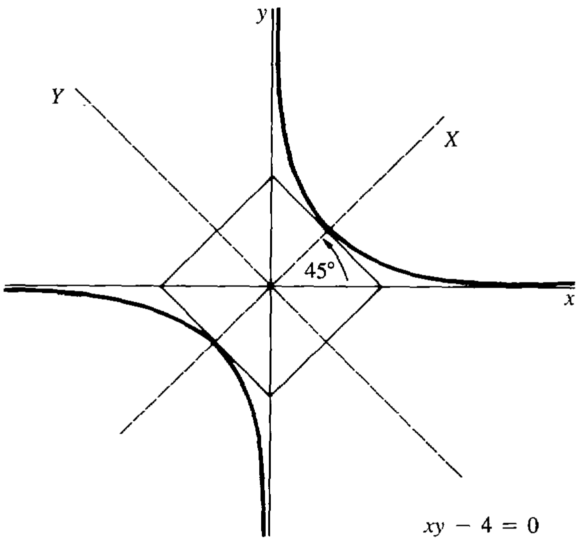


Figure 5.7.5 Example 2

Step 2 *Find an angle α with*

$$\cot(2\alpha) = \frac{A - C}{B}.$$

Step 3 *Change coordinate axes using the Rotation Equations. The new equation has the form*

$$A_1X^2 + C_1Y^2 + D_1X + E_1Y + F_1 = 0,$$

where $\quad x = X\cos\alpha - Y\sin\alpha, \qquad y = X\sin\alpha + Y\cos\alpha.$

Step 4 *Draw the X- and Y-axes by rotating the old axes through the angle α. The curve can now be sketched by our previous method, using Translation of Axes if necessary.*

Here is an overall summary of the use of rotations and translations of axes. The problem is to graph an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

By Rotation of Axes, we get a new equation of the simpler form

$$A_1X^2 + C_1Y^2 + D_1X + E_1Y + F_1 = 0.$$

If either $A_1 = 0$ or $C_1 = 0$, the curve is a parabola that can be sketched by the method of Section 5.4. If A_1 and C_1 are both nonzero, Translation of Axes gives us a new equation of the simpler form

$$A_2U^2 + B_2V^2 + F_2 = 0.$$

The graph of this equation is an ellipse or hyperbola, which can be sketched by the method of Section 5.5. The degenerate cases—two lines, one line, a point, or an empty graph—may also occur.

PROBLEMS FOR SECTION 5.7

In Problems 1–10, rotate the axes to transform the given equation into a new equation with no XY-term. Find the angle of rotation and the new equation.

1 $xy + 4 = 0$

2 $x^2 + xy + y^2 = 2$

3 $x^2 - 4xy + y^2 = 1$

4 $x^2 + 3xy + y^2 = 4$

5 $x^2 + 2\sqrt{3}xy - y^2 = 7$

6 $5x^2 - \sqrt{3}xy + 4y^2 = 6$

7 $x^2 + xy = 3$

8 $2x^2 - xy - y^2 = 1$

9 $4x^2 - \sqrt{3}xy + y^2 = 5$

10 $2x^2 + \sqrt{3}xy - y^2 = -10$

11 Prove that any second degree Equation (1) in which $A = C$ can be transformed into an equation with no XY-term by a 45° rotation of axes.

☐ 12 Prove that if we begin with a second degree equation with no first degree terms, $Ax^2 + Bxy + Cy^2 + F = 0$, and then rotate axes, the new equation will again have no first degree terms.

☐ 13 Prove that the sum $A + C$ is not changed by rotation of axes. That is, if Equation (2) is obtained from Equation (1) by rotation of axes, then $A + C = A_1 + C_1$.

☐ 14 Prove that the discriminant of a second degree equation is not changed by rotation of axes. That is, if Equation (2) is obtained from Equation (1) by rotation of axes, then $B^2 - 4AC = B_1^2 - 4A_1C_1$.

5.8 THE ε, δ CONDITION FOR LIMITS

The traditional calculus course is developed entirely without infinitesimals. The starting point is the concept of a limit. The intuitive idea of $\lim_{x \to c} f(x) = L$ is: *For every real number x which is close to but not equal to c, $f(x)$ is close to L.*

It is hard to make this idea into a rigorous definition, because one must clarify the word "close". Indeed, the whole point of our infinitesimal approach to calculus is that it is easier to define and explain limits using infinitesimals. The definition of limits in terms of real numbers is traditionally expressed using the Greek letters ε (epsilon) and δ (delta), and is therefore called the ε, δ condition for limits.

The ε, δ condition will be based on the notion of distance between two real numbers.

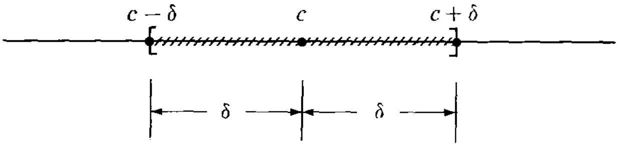
DEFINITION

*The **distance** between two real numbers x and c is the absolute value of their difference,*

$$\textit{distance} = |x - c|.$$

x is within δ of c if $|x - c| \le \delta$.
x is strictly within δ of c if $|x - c| < \delta$.

Notice that the distance $|x - c|$ is just the difference between the larger and the smaller of the two numbers x and c. This is a place where the absolute value sign is especially convenient. The following simple but helpful lemma is illustrated in Figure 5.8.1.



(a) x within δ of c

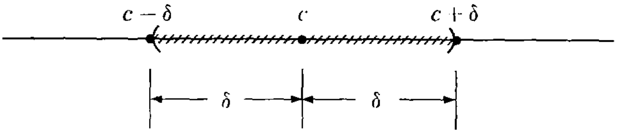


Figure 5.8.1 (b) x strictly within δ of c

LEMMA

(i) *x is within δ of c if and only if*

$$c-\delta \le x \le c+\delta.$$

(ii) *x is strictly within δ of c if and only if*

$$c-\delta < x < c+\delta.$$

PROOF (i) Subtracting c from each term we see that

$$c-\delta \le x \le c+\delta$$

if and only if $$-\delta \le x-c \le \delta,$$

which is true if and only if $|x-c| \le \delta$.

The proof of (ii) is similar.

We shall repeat our infinitesimal definition of limit from Section 3.3 and then write down the ε, δ condition for limits. Later we shall prove that the two definitions of limit are equivalent to each other.

Suppose the real function f is defined for all real numbers $x \ne c$ in some neighborhood of c.

DEFINITION OF LIMIT (Repeated)

The equation

$$\lim_{x\to c} f(x) = L$$

means that whenever a hyperreal number x is infinitely close to but not equal to c, $f(x)$ is infinitely close to L.

ε, δ CONDITION FOR $\lim_{x\to c} f(x) = L$

For every real number $\varepsilon > 0$ there is a real number $\delta > 0$ which depends on ε such that whenever x is strictly within δ of c but not equal to c, $f(x)$ is strictly within ε of L. In symbols, if $0 < |x-c| < \delta$, then $|f(x)-L| < \varepsilon$.

In the ε, δ condition, the notion of being infinitely close to c is replaced by being strictly within δ of c, and being infinitely close to L is replaced by being strictly within ε of L. But why are there two numbers ε and δ, instead of just one? And why should δ depend on ε? Let us look at a simple example.

EXAMPLE 1 Consider the limit $\displaystyle\lim_{x\to 0}\left(1+\frac{10x^2}{x}\right)=1.$

When $x = 0$, the function $f(x) = 1 + 10x^2/x$ is undefined. When x is a real number close to but not equal to 0, $f(x)$ is close to 1.

Now let us be more explicit. How should we choose x to get $f(x)$ strictly within $\frac{1}{5}$ of 1? To solve this problem we assume x is strictly within some distance δ of 0 and get inequalities for $f(x)$.

By the lemma, we must find a $\delta > 0$ such that whenever

$$-\delta < x < \delta \quad \text{and} \quad x \neq 0,$$

we have
$$1-\tfrac{1}{5} < f(x) < 1+\tfrac{1}{5}.$$

$$\begin{array}{llcll}
\text{Assume} & -\delta < x & \text{and} & x < \delta. & \\
\text{Then} & -10\delta < 10x & \text{and} & 10x < 10\delta & \\
 & -10\delta < \dfrac{10x^2}{x} & \text{and} & \dfrac{10x^2}{x} < 10\delta & \text{if } x \neq 0 \\
 & 1-10\delta < 1+\dfrac{10x^2}{x} & \text{and} & 1+\dfrac{10x^2}{x} < 1+10\delta &
\end{array}$$

$$1-10\delta < f(x) < 1+10\delta.$$

If we set $\delta = \frac{1}{50}$, then

$$1-\tfrac{1}{5} < f(x) < 1+\tfrac{1}{5}.$$

This shows that

$$\text{whenever } -\tfrac{1}{50} < x < \tfrac{1}{50} \text{ and } x \neq 0, \qquad 1-\tfrac{1}{5} < f(x) < 1+\tfrac{1}{5}.$$

In other words,

$$\text{whenever } 0 < |x| < \tfrac{1}{50}, \qquad |f(x)-1| < \tfrac{1}{5}.$$

A similar computation shows that for each $\varepsilon > 0$, if $0 < |x| < \varepsilon/10$ then $|f(x) - 1| < \varepsilon$. Thus the ε, δ condition for $\lim_{x\to 0}(1 + 10x^2/x) = 1$ is true, and, for a given ε, a corresponding δ is $\delta = \varepsilon/10$.

EXAMPLE 2 In the limit

$$\lim_{x\to 2} x^2 = 4,$$

find a $\delta > 0$ such that whenever $0 < |x - 2| < \delta$, $|x^2 - 4| < \frac{1}{10}$.

By the Lemma, we must find $\delta > 0$ such that whenever

$$2-\delta < x < 2+\delta \quad \text{and} \quad x \neq 2,$$
$$4-\tfrac{1}{10} < x^2 < 4+\tfrac{1}{10}.$$

Assume that $\qquad 2-\delta < x$ and $x < 2+\delta$.

As long as $2 - \delta$ and x are positive we may square both sides,

$$4-4\delta+\delta^2 < x^2 \quad \text{and} \quad x^2 < 4+4\delta+\delta^2$$

$$4 + (-4\delta + \delta^2) < x^2 \quad \text{and} \quad x^2 < 4 + (4\delta + \delta^2).$$

Now take δ small enough so that

$$-\tfrac{1}{10} \le -4\delta + \delta^2 \quad \text{and} \quad 4\delta + \delta^2 \le \tfrac{1}{10}.$$

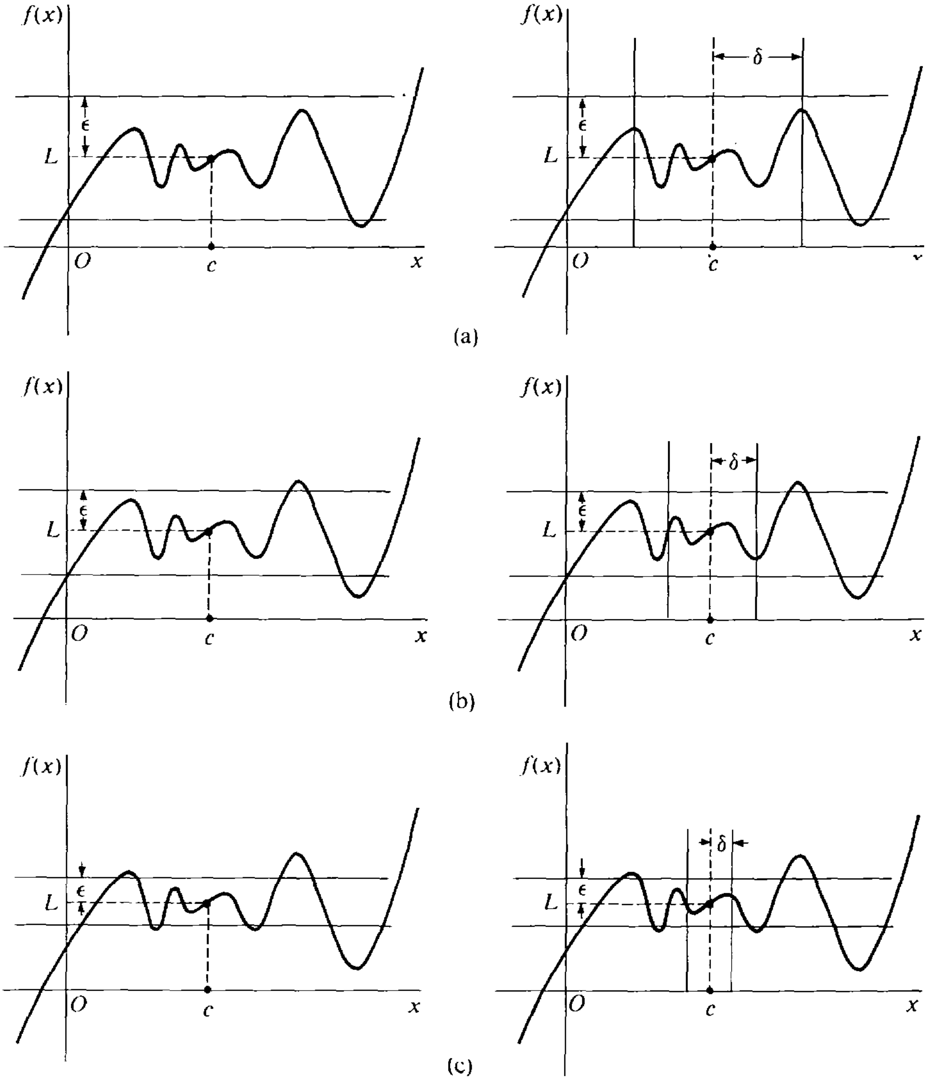
For example, $\delta = \frac{1}{50}$ will do. Then

$$4 - \tfrac{1}{10} < x^2 < 4 + \tfrac{1}{10}.$$

Thus whenever $0 < |x - 2| < \frac{1}{50}$, $\quad |x^2 - 4| < \frac{1}{10}$.

Notice that any smaller value of δ, such as $\delta = \frac{1}{100}$, will also work.

In geometric terms, the ε, δ condition says that for every horizontal strip (of width 2ε) centered at L, there exists a vertical strip (of width 2δ) centered at c such that whenever $x \neq c$ is in the vertical strip, $f(x)$ is in the horizontal strip. The graphs in Figure 5.8.2 indicate various horizontal strips and corresponding vertical strips. They should be examined closely.



(a)

(b)

(c)

Figure 5.8.2

There are also ε, δ conditions for one-sided limits and infinite limits. The three cases below are typical.

ε, δ CONDITION FOR $\lim_{x \to c^+} f(x) = L$

For every real number $\varepsilon > 0$, there is a real number $\delta > 0$ which depends on ε such that whenever $c < x < c + \delta$, we have $|f(x) - L| < \varepsilon$.

Intuitively, when x is close to c but greater than c, $f(x)$ is close to L.

ε, δ CONDITION FOR $\lim_{x \to \infty} f(x) = L$

For every real number $\varepsilon > 0$ there is a real number $B > 0$ which depends on ε such that whenever $x > B$, we have $|f(x) - L| < \varepsilon$.

Intuitively, when x is large, $f(x)$ is close to L.

ε, δ CONDITION FOR $\lim_{x \to \infty} f(x) = \infty$

For every real number $A > 0$ there is a real number $B > 0$ which depends on A such that whenever $x > B$, we have $f(x) > A$.

Intuitively, when x is large, $f(x)$ is large.

EXAMPLE 3 In the limit

$$\lim_{t \to \infty} 2 + \frac{3}{t} = 2,$$

find a real number $B > 0$ such that whenever $t > B$, $(2 + 3/t)$ is strictly within 1/100 of 2.

To find B, we assume $t > B$ and $t > 0$, and get inequalities for $2 + 3/t$.

$$\begin{array}{ll} 0 < t, & t > B \\ 0 < \dfrac{3}{t}, & \dfrac{3}{t} < \dfrac{3}{B} \\ 2 < 2 + \dfrac{3}{t}, & 2 + \dfrac{3}{t} < 2 + \dfrac{3}{B} \end{array}$$

Now choose B so that $3/B \le 1/100$. The number $B = 300$ will do. It follows that whenever $t > 300$,

$$2 < 2 + \frac{3}{t} < 2 + \frac{1}{100},$$

and $2 + \dfrac{3}{t}$ is strictly within $\varepsilon = \dfrac{1}{100}$ of 2.

EXAMPLE 4 In the limit

$$\lim_{x \to \infty} (x^2 - x) = \infty.$$

find a $B > 0$ such that whenever $x > B$, $x^2 - x > 10{,}000$.

This time we assume $x > B$ and get an inequality for $x^2 - x$. We may assume $B > 1$.

$$\begin{aligned} x &> B > 1 \\ x - 1 &> B - 1 > 0 \\ x(x-1) &> B(B-1) \\ x^2 - x &> B^2 - B. \end{aligned}$$

Now take a B such that $B^2 - B > 10{,}000$. The number $B = 200$ will do, because $(200)^2 - 200 = 39800$. Thus whenever $x > 200$, $x^2 - x > 10{,}000$.

We conclude this section with the proof that the ε, δ condition is equivalent to the infinitesimal definition of a limit.

THEOREM 1

Let f be defined in some deleted neighborhood of c. Then the following are equivalent:

(i) $\lim_{x \to c} f(x) = L$.

(ii) *The ε, δ condition for $\lim_{x \to c} f(x) = L$ is true.*

PROOF We first assume the ε, δ condition and prove that

$$\lim_{x \to c} f(x) = L.$$

Let x be any hyperreal number which is infinitely close but not equal to c. To prove that $f(x)$ is infinitely close to L we must show that

$$\text{for every real } \varepsilon > 0, \qquad |f(x) - L| < \varepsilon.$$

Let ε be any positive real number, and let $\delta > 0$ be the corresponding number in the ε, δ condition. Since x is infinitely close to c and $\delta > 0$ is real, we have

$$0 < |x - c| < \delta.$$

By the ε, δ condition and the Transfer Principle,

$$|f(x) - L| < \varepsilon.$$

We conclude that $f(x)$ is infinitely close to L. This proves that

$$\lim_{x \to c} f(x) = L.$$

For the other half of the proof we assume that

$$\lim_{x \to c} f(x) = L,$$

and prove the ε, δ condition. This will be done by an indirect proof. Assume that the ε, δ condition is false for some real number $\varepsilon > 0$. That means that for every real $\delta > 0$ there is a real number $x = x(\delta)$ such that

(1) $$x \neq c, \qquad |x - c| < \delta, \qquad |f(x) - L| \geq \varepsilon.$$

Now let $\delta_1 > 0$ be a positive infinitesimal. By the Transfer Principle, Equation (1) holds for δ_1. Therefore $x_1 = x(\delta_1)$ is infinitely close but not equal to c. But since

$$|f(x_1) - L| \geq \varepsilon$$

and ε is a positive real number, $f(x_1)$ is not infinitely close to L. This contradicts the equation

$$\lim_{x \to c} f(x) = L.$$

We conclude that the ε, δ condition must be true after all.

The theorem is also true for the other types of limits.

The concept of continuity can be described in terms of limits, as we saw in Section 3.4. Therefore continuity can be defined in terms of the real number system only.

COROLLARY

The following are equivalent.

(i) *f is continuous at c.*

(ii) *For every real $\varepsilon > 0$ there is a real $\delta > 0$ depending on ε such that:*

$$\textit{whenever} \quad |x - c| < \delta, \qquad |f(x) - f(c)| < \varepsilon.$$

PROOF Both (i) and (ii) are equivalent to

$$\lim_{x \to c} f(x) = f(c).$$

Intuitively, this corollary says that *f is continuous at c* if and only if *$f(x)$ is close to $f(c)$ whenever x is close to c.*

PROBLEMS FOR SECTION 5.8

1 In the limit $\lim_{x \to 4} 10x = 40$, find a $\delta > 0$ such that whenever $0 < |x - 4| < \delta$, $|10x - 40| < 0.01$.

2 In the limit $\lim_{x \to 0} (x^2 - 4x)/2x = -2$, find a $\delta > 0$ such that whenever $0 < |x| < \delta$, $|(x^2 - 4x)/2x - (-2)| < 0.1$.

3 In the limit $\lim_{x \to 2} 1/x = 1/2$, find a $\delta > 0$ such that whenever $0 < |x - 2| < \delta$, $|1/x - 1/2| < 0.01$.

4 In the limit $\lim_{x \to -3} x^3 = -27$, find a $\delta > 0$ such that whenever $0 < |x - (-3)| < \delta$, $|x^3 - (-27)| < 0.01$.

5 In the limit $\lim_{x \to 0^+} \sqrt{x} = 0$, find a $\delta > 0$ such that whenever $0 < x < \delta$, $\sqrt{x} < 0.01$.

6 In the limit $\lim_{x \to 2^+} \sqrt{x^2 - 4} = 0$, find a $\delta > 0$ such that whenever $2 < x < 2 + \delta$, $\sqrt{x^2 - 4} < 0.1$.

7 In the limit $\lim_{x \to 1^-} \sqrt{1 - x^2} = 0$, find a $\delta > 0$ such that whenever $1 - \delta < x < 1$, $\sqrt{1 - x^2} < 0.001$.

8 In the limit $\lim_{x \to 2^-} \sqrt{6 - 3x} = 0$, find a $\delta > 0$ such that whenever $2 - \delta < x < 2$, $\sqrt{6 - 3x} < 0.01$.

9 In the limit $\lim_{x \to 0} x^{-2} = \infty$, find a $\delta > 0$ such that whenever $0 < |x| < \delta, x^{-2} > 10{,}000$.

10 In the limit $\lim_{x\to 0} 16/x^4 = \infty$, find a $\delta > 0$ such that whenever $0 < |x| < \delta$, $16/x^4 > 10{,}000$.

11 In the limit $\lim_{t\to 0^+} 1/10t = \infty$, find a $\delta > 0$ such that whenever $0 < t < \delta$, $1/10t > 100$.

12 In the limit $\lim_{t\to 4^+} 1/(4 - t) = -\infty$, find a $\delta > 0$ such that whenever $4 < t < 4 + \delta$, $1/(4 - t) < -100$.

13 In the limit $\lim_{x\to 0^+} 1/\sqrt{x} = \infty$, find a $\delta > 0$ such that whenever $0 < x < \delta$, $1/\sqrt{x} > 100$.

14 In the limit $\lim_{x\to 0^+} 1/x^3 = \infty$, find a $\delta > 0$ such that whenever $0 < x < \delta$, $1/x^3 > 1000$.

15 In the limit $\lim_{x\to 1^-} 1/(1 - x^2) = \infty$, find a $\delta > 0$ such that whenever $1 - \delta < x < 1$, $1/(1 - x^2) > 100$.

16 In the limit $\lim_{x\to 2^-} 5/\sqrt{2 - x} = \infty$, find a $\delta > 0$ such that whenever $2 - \delta < x < 2$, $5/\sqrt{2 - x} > 100$.

17 In the limit $\lim_{t\to\infty} 1/(1 + 4t) = 0$, find a $B > 0$ such that whenever $t > B$, $1/(1 + 4t) < 0.01$.

18 In the limit $\lim_{t\to\infty} 1/t^2 = 0$, find a $B > 0$ such that whenever $t > B$, $1/t^2 < 0.01$.

19 In the limit $\lim_{t\to\infty} 2t^2 - 5t = \infty$, find a $B > 0$ such that whenever $t > B$, $2t^2 - 5t > 1000$.

20 In the limit $\lim_{t\to\infty} t^3 + t^2 - 5 = \infty$, find a $B > 0$ such that whenever $t > B$, $t^3 + t^2 - 5 > 1000$.

21 In the limit $\lim_{x\to\infty} \sqrt{5x + 1} = \infty$, find a $B > 0$ such that whenever $x > B$, $\sqrt{5x + 1} > 100$.

22 In the limit $\lim_{x\to -\infty} \sqrt[3]{x - 1} = -\infty$, find a $B > 0$ such that whenever $x < -B$, $\sqrt[3]{x - 1} < -100$.

☐ 23 State the ε, δ condition for the limit $\lim_{x\to c^-} f(x) = L$.

☐ 24 State the ε, δ condition for the limit $\lim_{x\to c} f(x) = \infty$.

☐ 25 State the ε, δ condition for the limit $\lim_{x\to\infty} f(x) = -\infty$.

☐ 26 Prove that $\lim_{x\to\infty} f(x) = \infty$ if and only if the ε, δ condition for this limit holds: For every $A > 0$ there is a $B > 0$ such that whenever $x > B$, $f(x) > A$.

5.9 NEWTON'S METHOD

The Increment Theorem for derivatives shows that when $f'(c)$ exists and $x \approx c$, $f(x)$ is infinitely close to the tangent line $f(c) + f'(c)(x - c)$ even compared to $x - c$. Thus intuitively, when x is real and close to c, $f(x)$ is closely approximated by the tangent line $f(c) + f'(c)(x - c)$. Newton's method uses the tangent line to approximate a zero of $f(x)$. It is an iterative method that does not always work but usually gives a very good approximation.

Consider a real function f that crosses the x-axis as in Figure 5.9.1. From the graph we make a first rough approximation x_1 to the zero of $f(x)$. To get a better approximation, we take the tangent line at x_1 and compute the point x_2 where the tangent line intersects the x-axis. At x_2, the curve $f(x)$ is very close to zero, so we take x_2 as our new approximation. The tangent line has the equation

$$y = f(x_1) + f'(x_1)(x - x_1).$$

We get a formula for x_2 by setting $y = 0$ and $x = x_2$ and then solving for x_2.

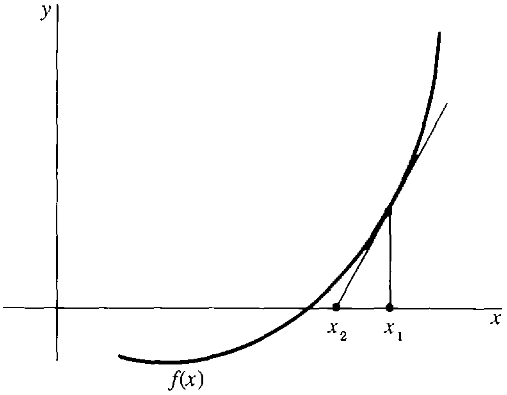


Figure 5.9.1

$$0 = f(x_1) + f'(x_1)(x_2 - x_1)$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

We may then repeat the procedure starting from x_2 to get a still better approximation x_3 as in Figure 5.9.2,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}.$$

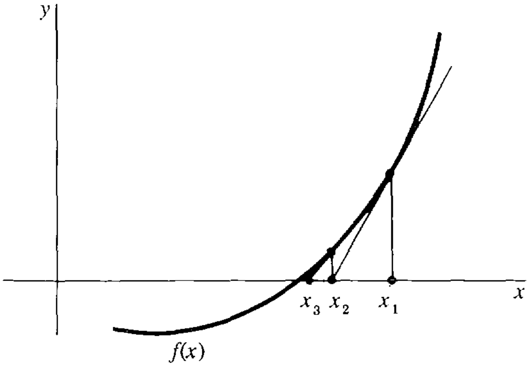


Figure 5.9.2

NEWTON'S METHOD

When to Use *We wish to approximate a zero of $f(x)$, where $f'(x)$ is continuous and not close to zero, as in Figure 5.9.1.*

Step 1 *Sketch the graph of $f(x)$, and choose a point x_1 near the zero of $f(x)$. x_1 is the first approximation.*

Step 2 *Compute $f'(x)$.*

Step 3 *Compute the second approximation*

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Step 4 *For a closer approximation repeat Step 3. The $(n + 1)$st approximation is given by*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

As a rough check on the accuracy, compute $f(x_n)$ and note how close it is to zero.

Steps 3 and 4 can be done conveniently on a hand calculator.

Warning: Since Newton's method involves division by $f'(x_1)$, avoid starting at a point where the slope is near zero. Figure 5.9.3 shows that when the slope is close to zero, the tangent line is nearly horizontal and the approximation may be poor.

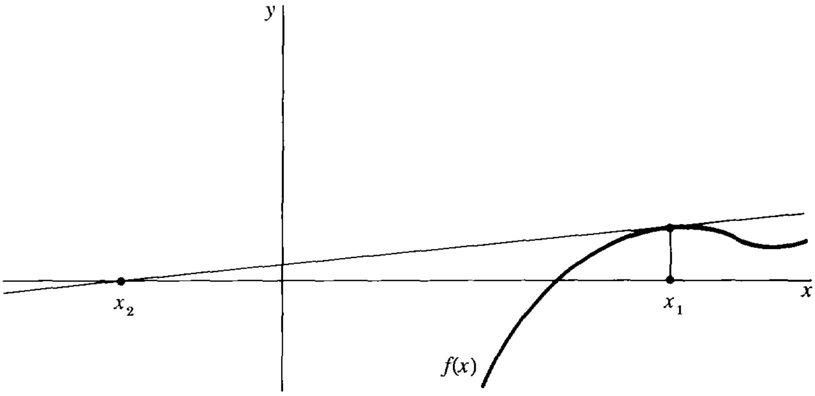


Figure 5.9.3

EXAMPLE 1 Approximate a zero of $f(x) = x^3 + 2x^2 - 5$ by Newton's method.

Step 1 The graph is shown in Figure 5.9.4. We choose $x_1 = 1$ as our first approximation.

Step 2 $f'(x) = 3x^2 + 4x$

Step 3 $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{(-2)}{7} = \frac{9}{7} \sim 1.2857$

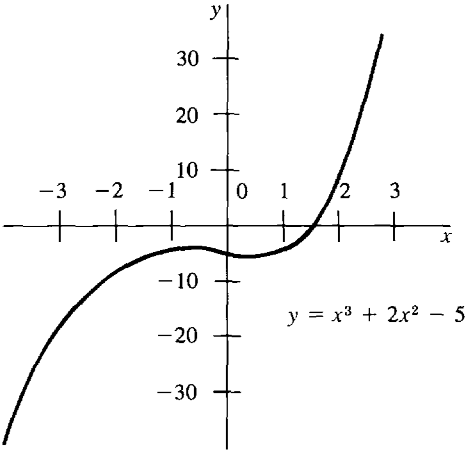


Figure 5.9.4

Step 4 $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = x_2 - \dfrac{x_2^3 + 2x_2^2 - 5}{3x_2^2 + 4x_2} \sim 1.2430$

As a check we compute

$$f(x_3) = x_3^3 + 2x_3^2 - 5 \sim 0.01$$

One more iteration gives much more accuracy:

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = x_3 - \frac{x_3^3 + 2x_3^2 - 5}{3x_3^2 + 4x_3} \sim 1.241897$$

$$f(x_4) = x_4^3 + 2x_4^2 - 5 \sim 0.000007$$

EXAMPLE 2 Approximate the fifth root of 6 by Newton's method.

Step 1 We must find the zero of $f(x) = x^5 - 6$. The graph is shown in Figure 5.9.5. Choose $x_1 = 1.5$.

Step 2 $f'(x) = 5x^4$

Step 3 $x_2 = x_1 - \dfrac{x_1^5 - 6}{5x_1^4} \sim 1.437$

Step 4 $x_3 = x_2 - \dfrac{x_2^5 - 6}{5x_2^4} \sim 1.43102$

As a check we compute

$$(x_3)^5 \sim 6.001$$

In this example more iterations would be necessary if our first approximation had not been chosen as well. For instance, starting with $x_1 = 1$ we would not reach

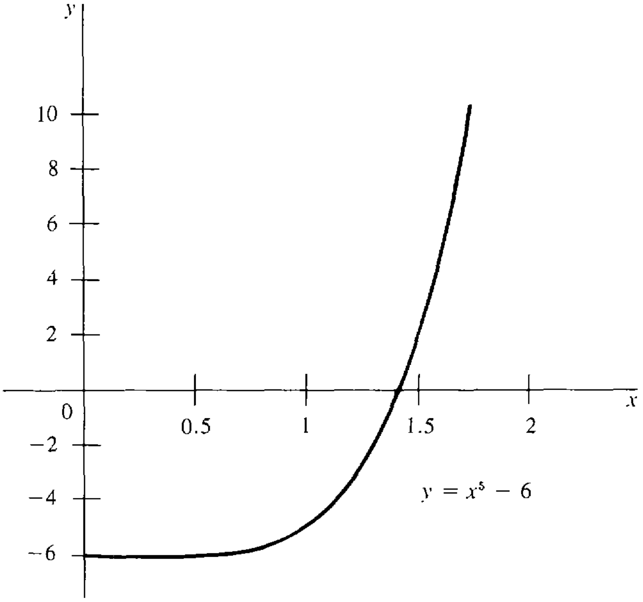


Figure 5.9.5

the approximation 1.431 until x_6, obtaining the successive approximations

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 1.675, \quad x_4 = 1.49245,$$
$$x_5 = 1.43583, \quad x_6 = 1.43100.$$

EXAMPLE 3 Approximate the point x where $\sin x = \ln x$.

As one can see from the graphs of $\sin x$ and $\ln x$ in Figure 5.9.6, $\sin x$ and $\ln x$ cross at one point x, which is somewhere between $x = 1$ (where $\ln x$ crosses the x-axis going up) and $x = \pi$ (where $\sin x$ crosses the x-axis going down). To apply Newton's method, we let $f(x)$ be the function

$$f(x) = \sin x - \ln x$$

shown in Figure 5.9.7. We wish to approximate the zero of $f(x)$.

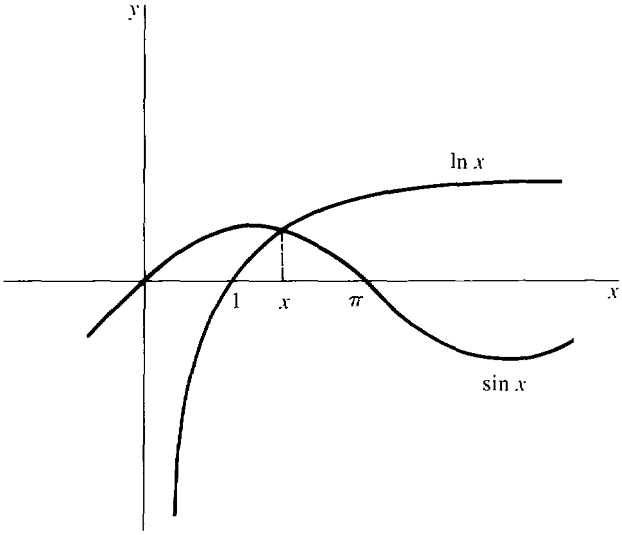


Figure 5.9.6

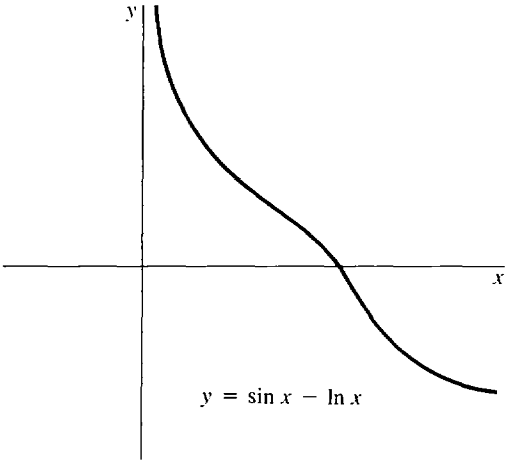


Figure 5.9.7

Step 1 Choose $x_1 = 2$ (since the zero of $f(x)$ is between 1 and π).

Step 2 $f'(x) = \cos x - 1/x$

Step 3 $$x_2 = x_1 - \frac{\sin x_1 - \ln x_1}{\cos x_1 - 1/x_1} = 2 - \frac{\sin 2 - \ln 2}{\cos 2 - 1/2} \sim 2.23593$$

Step 4 Repeat Step 3. The values of x_n, $f(x_n)$, and $f'(x_n)$ are shown in the table.

n	x_n	$f(x_n)$	$f'(x_n)$
1	2.000000000	0.216150246	−0.916146836
2	2.235934064	−0.017827280	−1.064407894
3	2.219185522	−0.000082645	−1.054519059
4	2.219107150	−0.000000001	−1.054472505

The answer is

$$x \sim 2.219107150.$$

On a calculator we find that

$$\sin(2.219107150) = 0.797104929$$
$$\ln(2.219107150) = 0.797104930.$$

PROBLEMS FOR SECTION 5.9

Use Newton's method to find approximate solutions to each of the following equations. (A hand calculator is recommended.)

1 $x^3 + 5x - 10 = 0$

2 $2x^3 + x + 4 = 0$

3 $x^5 + x^3 + x = 1$

4 $2x^5 + 3x = 2$

5 $x^4 = x + 1, \quad x > 0$

6 $x^4 = x + 1, \quad x < 0$

7 $x^3 - 10x + 4 = 0, \quad x > 1$

8 $x^3 - 10x + 4 = 0, \quad 0 < x < 1$

9 $x + \sqrt{x} = 1$

10 $x + 1/\sqrt{x} = 3$

11 $e^x = 1/x$

12 $e^x + x = 4$

13 $x + \sin x = 2$

14 $\cos x = x^2, \quad x > 0$

15 $\tan x = e^x, \quad 0 < x < \pi/2$

16 $e^x + \ln x = 0$

5.10 DERIVATIVES AND INCREMENTS

In Section 3.3 we found that the derivative of f is given by the limit

$$f'(c) = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

If $y = f(x)$, $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

By definition this means that when the hyperreal number Δx is infinitely close to but not equal to zero, $\Delta y/\Delta x$ is infinitely close to dy/dx.

By contrast, the ε, δ condition for this limit says intuitively that when the real number Δx is close to but not equal to zero, $\Delta y/\Delta x$ is close to dy/dx.

The ε, δ condition for the derivative can be given a geometric interpretation, shown in Figure 5.10.1. Consider the curve $y = f(x)$, and suppose $f'(c)$ exists. Draw the line tangent to the curve at c. For $\Delta x \neq 0$, draw the *secant line* which intersects the curve at the points $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$. Then the tangent line will have slope $f'(c)$ while the secant line will have slope

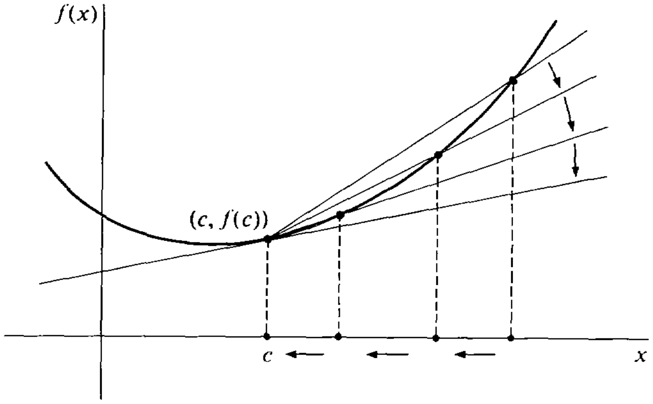


Figure 5.10.1

$$\frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

The ε, δ condition shows that if we take values of Δx closer and closer to zero, then the slopes of the secant line will get closer and closer to the slope of the tangent line.

EXAMPLE 1 Consider the curve $f(x) = x^{1/3}$.

Then $f'(x) = \frac{1}{3}x^{-2/3}$.

At the point $x = 8$, we have

$$x = 8, \qquad f(x) = 2, \qquad f'(x) = \tfrac{1}{12} = 0.0833\ldots.$$

Thus
$$\lim_{\Delta x \to 0} \frac{(8 + \Delta x)^{1/3} - 2}{\Delta x} = \frac{1}{12}.$$

This is the slope of the line tangent to the curve at the point $(8, 2)$. As Δx approaches zero, the slope of the secant line through the two points $(8, 2)$ and $(8 + \Delta x, (8 + \Delta x)^{1/3})$ will approach $\frac{1}{12}$. We make a table showing the slope of the secant line for various values of Δx.

Δx	$\Delta y = (8 + \Delta x)^{1/3} - 2$	$\dfrac{\Delta y}{\Delta x}$ = slope of secant line	$\left\lvert \dfrac{\Delta y}{\Delta x} - \dfrac{1}{12} \right\rvert$
10	0.6207	0.0621	0.0212
1	0.0801	0.0801	0.0032
$\frac{1}{10}$	0.00829	0.0830	0.0003
-10	-3.2599	0.3260	0.2427
-1	-0.0871	0.0871	0.0038
$-\frac{1}{10}$	-0.00837	0.0837	0.0004

The ε, δ condition for the derivative is of theoretical importance but does not give an error estimate for the limit. When the function f has a continuous second derivative, we can get a useful error estimate in a different way. It is more convenient to work with one-sided limits.

By an *error estimate* for a limit

$$\lim_{\Delta x \to 0^+} g(\Delta x) = L$$

we mean a real function $E(\Delta x)$, $0 < \Delta x \leq b$, such that the approximation $g(\Delta x)$ is always within $E(\Delta x)$ of the limit L. In symbols,

$$|g(\Delta x) - L| \leq E(\Delta x) \qquad \text{for} \quad 0 < \Delta x \leq b.$$

THEOREM 1

Suppose f has a continuous second derivative and $|f''(t)| \leq M$ for all t in the interval $[c, b]$. Then:

(i) *Whenever $c < c + \Delta x \leq b$, $f(c + \Delta x)$ is within $\frac{1}{2}M\,\Delta x^2$ of $f(c) + f'(c)\,\Delta x$.*

(ii) *Whenever $c < c + \Delta x \leq b$, $\dfrac{f(c + \Delta x) - f(c)}{\Delta x}$ is within $\frac{1}{2}M\,\Delta x$ of $f'(c)$. That is, $\frac{1}{2}M\,\Delta x$ is an error estimate for the right-sided limit*

$$\lim_{\Delta x \to 0^+} \frac{f(c + \Delta x) - f(c)}{\Delta x} = f'(c).$$

There is a similar theorem for the left-sided limit

$$\lim_{\Delta x \to 0^-} \frac{f(c + \Delta x) - f(x)}{\Delta x} = f'(c)$$

with the error estimate $\frac{1}{2}M|\Delta x|$.

PROOF Let $x = c + \Delta x$. Then

$$-M \leq f''(t) \leq M \qquad \text{for } c \leq t \leq x.$$

Integrating from c to t,

$$\int_c^t -M\,dt \leq \int_c^t f''(t)\,dt \leq \int_c^t M\,dt,$$

$$-M(t - c) \leq f'(t) - f'(c) \leq M(t - c).$$

Integrating again from c to x,

$$\int_c^x -M(t - c)\,dt \leq \int_c^x f'(t) - f'(c)\,dt \leq \int_c^x M(t - c)\,dt,$$

$$-M\frac{(x - c)^2}{2} \leq f(t) - f'(c)t\Big]_c^x \leq M\frac{(x - c)^2}{2},$$

or

$$-M\frac{\Delta x^2}{2} \leq (f(x) - f(c)) - f'(c)\,\Delta x \leq M\frac{\Delta x^2}{2},$$

$$-M\frac{\Delta x^2}{2} \leq f(x) - (f(c) + f'(c)\,\Delta x) \leq M\frac{\Delta x^2}{2}.$$

This proves part (i) (Figure 5.10.2). Dividing by Δx we get part (ii).

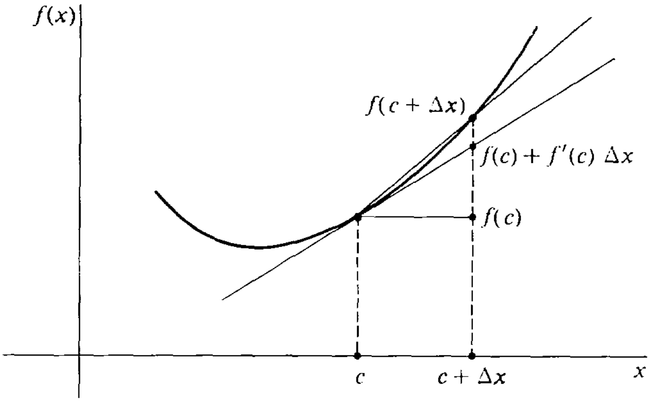


Figure 5.10.2

EXAMPLE 1 (Concluded) We consider once more the curve $f(x) = x^{1/3}$ at the point $x = 8$. The second derivative is

$$f''(x) = -\tfrac{2}{9}x^{-5/3}.$$

First consider the interval $[8, 9]$. In this interval $f''(x)$ has the maximum value

$$|f''(8)| = \tfrac{2}{9}(8)^{-5/3} = \tfrac{2}{9}2^{-5} = \tfrac{1}{144}.$$

Thus we may take $M = \frac{1}{144}$, and

$$\frac{1}{2}M\,\Delta x = \frac{1}{288}\Delta x \quad \text{is an error estimate for} \quad \lim_{\Delta x \to 0^+} \frac{\Delta y}{\Delta x} = \frac{1}{12}.$$

Thus

$$\text{when } \Delta x = 1, \qquad \left|\frac{\Delta y}{\Delta x} - \frac{1}{12}\right| \le \frac{1}{288} = 0.0035,$$

$$\text{when } \Delta x = \frac{1}{10}, \qquad \left|\frac{\Delta y}{\Delta x} - \frac{1}{12}\right| \le \frac{1}{2880} = 0.00035.$$

Next consider the interval $[7, 8]$. This time we take

$$M = |f''(7)| = \tfrac{2}{9}(7)^{-5/3} = 0.0087.$$

Then

$$\tfrac{1}{2}M|\Delta x| = 0.0044|\Delta x|$$

is an error estimate for the limit

$$\lim_{\Delta x \to 0^-} \frac{\Delta y}{\Delta x} = \frac{1}{12}.$$

$$\text{when } \Delta x = -1, \qquad \left|\frac{\Delta y}{\Delta x} - \frac{1}{12}\right| \le 0.0044,$$

$$\text{when } \Delta x = -\frac{1}{10}, \qquad \left|\frac{\Delta y}{\Delta x} - \frac{1}{12}\right| \le 0.00044.$$

From the table in Example 1 we see that the error estimates are slightly greater than the actual values of $\left|\dfrac{\Delta y}{\Delta x} - \dfrac{1}{12}\right|$.

We shall now turn the problem around. Instead of using the increment Δy to approximate the derivative dy/dx, we shall use the derivative dy/dx to approximate the increment Δy. When Δx is small, $f(c + \Delta x)$ will be close to $f(c) + f'(c)\,\Delta x$ even compared to Δx. Part (i) of Theorem 1 gives the error estimate $\frac{1}{2}M\,\Delta x^2$ for this approximation. This method is especially useful for approximating $f(x)$ when there is a number c close to x such that both $f(c)$ and $f'(c)$ are known.

EXAMPLE 2 Find approximate values for $\sqrt[3]{9}$ and $\sqrt[3]{7.9}$. Both these numbers are close to 8, whose cube root 2 comes out even. Taking $f(x) = \sqrt[3]{x}$ and $c = 8$, we have

$$f(c) = 2, \qquad f'(c) = \tfrac{1}{12} = 0.0833\ldots.$$

From Theorem 1 the approximate values are

$$f(c + \Delta x) \sim f(c) + f'(c)\,\Delta x.$$

Thus

$$\begin{aligned}\sqrt[3]{9} &\sim 2 + \tfrac{1}{12} = 2.0833,\\ \sqrt[3]{7.9} &\sim 2 + \tfrac{1}{12}(-0.1) = 1.99167.\end{aligned}$$

To get an error estimate for $\sqrt[3]{9}$, take the interval $[8, 9]$. From Example 1 we may take $M = \frac{1}{144}$. Therefore by Theorem 1,

$$\sqrt[3]{9} \sim 2.0833, \qquad \text{error} \le \tfrac{1}{2}\cdot\tfrac{1}{144}\cdot 1^2 = 0.0035.$$

Thus $$2.0798 \le \sqrt[3]{9} \le 2.0868.$$

To get an error estimate for $\sqrt[3]{7.9}$ take the interval $[7, 8]$ and $M = 0.0087$. By Theorem 1,

$$\sqrt[3]{7.9} \sim 1.991667, \qquad \text{error} \le \tfrac{1}{2}(0.0087)(0.1)^2 = 0.000044.$$

Thus $$1.991623 \le \sqrt[3]{7.9} \le 1.991711.$$

EXAMPLE 3 Find an approximate value for $(0.99)^5$.

Let $$f(x) = x^5, \quad c = 1.$$

Then $$f(c) = 1^5 = 1, \quad f'(c) = 5c^4 = 5.$$

We put $$0.99 = c + \Delta x, \qquad \Delta x = -0.01.$$

Then the approximate value is

$$\begin{aligned}f(c + \Delta x) &\sim f(c) + f'(c)\,\Delta x,\\ (0.99)^5 &\sim 1 + 5(-0.01) = 0.95.\end{aligned}$$

To get an error estimate we see that $f''(u) = 20u^3$, so $|f''(u)| \le 20$ for u between 0.99 and 1. Then $M = 20$, and

$$(0.99)^5 \sim 0.95, \qquad \text{error} \le \frac{(0.01)^2}{2}(20) = 0.001,$$

or $$0.949 \le (0.99)^5 \le 0.951.$$

Theorem 1 is closely related to the Increment Theorem in Section 2.2. The relation between them can be seen when we write them next to each other.

INCREMENT THEOREM (Repeated)

Hypotheses $f'(c)$ *exists and* Δx *is infinitesimal.*

Conclusion $f(c + \Delta x) = f(c) + f'(c)\,\Delta x + \varepsilon\,\Delta x$ *for some infinitesimal* ε *which depends on* c *and* Δx.

THEOREM 1 OF THIS SECTION (in an equivalent form)

Hypotheses $f''(u)$ *exists and* $|f''(u)| \le M$ *for all* u *between the real numbers* c *and* $c + \Delta x$.

Conclusion $f(c + \Delta x) = f(c) + f'(c)\,\Delta x + \varepsilon\,\Delta x$ *for some real* ε *within* $\frac{1}{2}M|\Delta x|$ *of* 0.

Thus Theorem 1 has more hypotheses but also gives more specific information about ε in its conclusion.

PROBLEMS FOR SECTION 5.10

In Problems 1–6, find $f'(c)$ and an error estimate for the limit

$$f'(c) = \lim_{\Delta x \to 0^+} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

with $0 < \Delta x \le 1$.

1 $f(x) = x^2, \quad c = 1$

2 $f(x) = x^3 - 5x, \quad c = 10$

3 $f(x) = 2/\sqrt{x}, \quad c = 4$

4 $f(x) = x\sqrt{x}, \quad c = 4$

5 $f(x) = 1/x, \quad c = 3$

6 $f(x) = 1/(x^2 + 1), \quad c = 1$

7 $f(x) = \sin x, \quad c = 0, \quad (0 < \Delta x \le \pi)$

8 $f(x) = \tan x, \quad c = 0, \quad (0 < \Delta x \le \pi/6)$

9 $f(x) = \cos(2x), \quad c = \pi/3, \quad (0 < \Delta x \le \pi)$

10 $f(x) = \sin^2(2x), \quad c = \pi/2, \quad (0 < \Delta x \le \pi)$

11 $f(x) = \ln x, \quad c = 1, \quad (0 < \Delta x \le 1)$

12 $f(x) = x \ln x, \quad c = 1, \quad (0 < \Delta x \le 1)$

13 $f(x) = e^x, \quad c = 1, \quad (0 < \Delta x \le 1)$

14 $f(x) = e^{x^2}, \quad c = 0, \quad (0 < \Delta x \le 1)$

In Problems 15–20, find $f'(c)$ and an error estimate for the limit

$$f'(c) = \lim_{\Delta x \to 0^-} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

with $-1 \le \Delta x < 0$.

15 $f(x) = \sqrt{x}, \quad c = 100$

16 $f(x) = 1/(3x + 6), \quad c = 0$

17 $f(x) = \sqrt{x^2 + 1}, \quad c = 2$

18 $f(x) = 4x^3, \quad c = 1$

19 $f(x) = x\sqrt{x + 1}, \quad c = 1$

20 $f(x) = x^{10}, \quad c = 2$

In Problems 21–38, approximate the given quantity and give an estimate of error.

21 $\sqrt{65}$

22 $1/\sqrt{50}$

23 $(0.301)^4$

24 $\sqrt[5]{30}$

25 $1/97$

26 $(99)^{3/2}$

27	$\sqrt{1.02} + \sqrt[3]{1.02}$	**28**	$(101 + \sqrt{101})^3$
29	$(1.003)^5$	**30**	$\sqrt[3]{0.9997}$
31	$\sin\left(\frac{\pi}{3} + 0.004\right)$	**32**	$\cos\left(\frac{\pi}{2} + 0.06\right)$
33	$\tan(0.005)$	**34**	$\sin(-0.003)$
35	$e^{0.002}$	**36**	$e^{-0.04}$
37	$\ln(1.006)$	**38**	$\ln(0.98)$

EXTRA PROBLEMS FOR CHAPTER 5

In Problems 1–10, find the limit.

1	$\lim\limits_{x\to\infty} \dfrac{2x^2 - 3x + 2}{x^3 + 5x^2 - 1}$	**2**	$\lim\limits_{x\to\infty} \dfrac{2x + 4}{5 - 3x}$
3	$\lim\limits_{x\to-\infty} x^{-1/3}$	**4**	$\lim\limits_{x\to\infty} (\sqrt[4]{x+1} - \sqrt[4]{x})x^{3/4}$
5	$\lim\limits_{x\to(2/3)^+} \dfrac{3x + 2}{3x - 2}$	6	$\lim\limits_{x\to1} \dfrac{x - 1}{\sqrt{x} - 1}$
7	$\lim\limits_{x\to1^+} \dfrac{x^{3/2} - 1}{\sqrt{x - 1}}$	**8**	$\lim\limits_{x\to4} \dfrac{x^2 + 3x - 1}{x^2 - 16}$
9	$\lim\limits_{x\to\infty} \dfrac{(x + 1)^{3/2} - x^{3/2}}{\sqrt{x}}$	**10**	$\lim\limits_{x\to\infty} \dfrac{x + \sqrt{x + 1}}{x - \sqrt{x + 1}}$

11 Sketch the curve $y = x - 1/x$.

12 Sketch the curve $y = 1 - x^{1/3}$.

13 Sketch the curve $y = 1/((x - 1)(x - 2))$.

14 Sketch the curve $y^2 - 4x^2 = 9$.

15 Sketch the curve $y = |x - 1| + |x + 1|$.

16 Find the equation of the parabola with directrix $y = 1$ and focus $F(1, -1)$.

17 Sketch the curve $y = -x^2 + 2x + 4$.

18 Sketch the curve $y = (\frac{1}{4})x^2 + x$.

19 Find the foci and sketch the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

20 Find the foci and sketch the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1.$$

21 Use Translation of Axes to sketch the curve

$$4x^2 + y^2 - 16x + 2y + 16 = 0.$$

22 Use Translation of Axes to sketch the curve

$$-x^2 + 4y^2 - 6x - 10 = 0.$$

23 Use Rotation of Axes to transform the equation $xy - 9 = 0$ into a second degree equation with no XY-term. Find the angle of rotation and the new equation.

24 Use Rotation of Axes to transform the equation $xy - y^2 = 5$ into a second degree equation with no XY-term. Find the angle of rotation and the new equation.

25 In the limit $\lim_{x\to4} 1/\sqrt{x} = 1/2$, find a $\delta > 0$ such that whenever $0 < |x - 4| < \delta$, $|1/\sqrt{x} - 1/2| < 0.01$.

26 In the limit $\lim_{x\to\infty}(x^2 - 1)^{1/2} = \infty$, find a $B > 0$ such that whenever $x > B$, $(x^2 - 1)^{1/2} > 10{,}000$.

27 Use Newton's method to find an approximate solution to the equation $x + x^{1/3} = 3$.

28 Use Newton's method to find an approximate solution to the equation $\cos x = \ln x$.

29 Find an error estimate for the limit

$$\lim_{\Delta x\to 0^+}\frac{(16 + \Delta x)^{1/4} - 2}{\Delta x} = \frac{1}{32}, \qquad 0 < \Delta x \le 1.$$

30 Find an error estimate for the limit

$$\lim_{\Delta x\to 0^+}\frac{(3 + \Delta x)^{-2} - \frac{1}{9}}{\Delta x} = -\frac{2}{27}, \qquad 0 < \Delta x \le 1.$$

31 Find an approximate value for $(124)^{2/3}$ and give an estimate of error.

32 Find an approximate value for $(0.9996)^6$ and give an estimate of error.

☐ 33 Prove that $\lim_{x\to\infty} f(x)$ exists if and only if whenever H and K are positive infinite, $f(H)$ is finite and $f(H) \approx f(K)$.

☐ 34 Prove that if $\lim_{t\to\infty} f(t) = L$ and $g(x)$ is continuous at $x = L$ then $\lim_{t\to\infty} g(f(t)) = g(L)$.

☐ 35 Prove that if $\lim_{t\to\infty} f(t) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ then $\lim_{t\to\infty} g(f(t)) = \infty$.

☐ 36 Suppose $\lim_{t\to\infty} f(t) = \infty$, c is a positive constant, and $cg(t) \ge f(t)$ for all t. Prove that $\lim_{t\to\infty} g(t) = \infty$.

☐ 37 Prove that $\lim_{x\to c} f(x) = L$ if and only if for every real $\varepsilon > 0$ there is a hyperreal $\delta > 0$ such that whenever $|x - c| < \delta$, $|f(x) - L| < \varepsilon$.

☐ 38 Let f be the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Using the ε, δ condition, prove that $f(x)$ is discontinuous at every real number $x = c$.

☐ 39 Let g be the function

$$g(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $g(x)$ is continuous at $x = 0$ but discontinuous everywhere else.

☐ 40 Prove that the function g in the preceding problem is not differentiable at $x = 0$.

☐ 41 Let

$$h(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that $h'(0)$ exists and equals 0.

☐ 42 Suppose $f(t)$ is continuous for all t and

$$\lim_{t\to -\infty} f(t) = A, \qquad \lim_{t\to\infty} f(t) = B.$$

If $A < C < B$, prove that there is a real number c with $f(c) = C$.